M. Q. Rieck
2/2/2022

Some claims concerning a tetrahedron based on an acute triangle

Consider a tetrahedron ABCP such that the triangle ABC is acute. Without altering its shape, scale and position this tetrahedron in Cartesian space so that the coordinates of A, B and C are $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$, respectively, with $x_j = \cos\phi_j$, $y_j = \sin\phi_j$ $(j = 1,2,3)$ and $\phi_1 + \phi_2 + \phi_3 = 0$ for angles ϕ_1, ϕ_2 and ϕ_3. (This is straight-forward to achieve.)

Define the following:

$$\angle A = \angle \mathrm{CAB},\ \angle B = \angle \mathrm{ABC},\ \angle C = \angle \mathrm{BCA},\ \alpha = \angle \mathrm{BPC},\ \beta = \angle \mathrm{CPA},\ \gamma = \angle \mathrm{APB},$$

$$C_0 = \cos\alpha\cos\beta\cos\gamma,\ C_1 = \cos^2\alpha,\ C_2 = \cos^2\beta,\ C_3 = \cos^2\gamma,$$

$$L = 2\,[\,(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)\,y_1\,(C_1 - 1) + (1 - x_2)\,y_2\,(C_2 - 1) + (1 - x_3)\,y_3\,(C_3 - 1)\,],$$

$$R = 2\,[\,(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)\,x_1\,(C_1 - 1) + (1 + x_2)\,x_2\,(C_2 - 1) + (1 + x_3)\,x_3\,(C_3 - 1)\,],$$

$$H = 1 + 2C_0 - C_1 - C_2 - C_3, \qquad E = L^2 + (R+H)^2,$$

$$D = E^2 + 18H^2E + 8H\,(R+H)\,[(R+H)^2 - 3L^2] - 27H^4.$$

The following (proven) constraints hold:

1. $H > 0$ (which is equivalent to $\alpha + \beta + \gamma < 2\pi \wedge \alpha < \beta + \gamma \wedge \beta < \gamma + \alpha \wedge \gamma < \alpha + \beta$);

2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$;

3. $\beta + \gamma - \alpha < 2(\angle B + \angle C)$, $\gamma + \alpha - \beta < 2(\angle C + \angle A)$, $\alpha + \beta - \gamma < 2(\angle \mathrm{A} + \angle B)$;

4. $\alpha < \angle A \rightarrow [\,\beta < \max\{\angle B, \angle C + \alpha\} \wedge \gamma < \max\{\angle C, \angle B + \alpha\}\,]$,
   $\beta < \angle B \rightarrow [\,\gamma < \max\{\angle C, \angle A + \beta\} \wedge \alpha < \max\{\angle A, \angle C + \beta\}\,]$,
   $\gamma < \angle C \rightarrow [\,\alpha < \max\{\angle A, \angle B + \gamma\} \wedge \beta < \max\{\angle B, \angle A + \gamma\}\,]$;

5. $\alpha < \angle A \rightarrow \cos\angle C\cos\beta + \cos\angle B\cos\gamma > 0$,
   $\beta < \angle B \rightarrow \cos\angle A\cos\gamma + \cos\angle C\cos\alpha > 0$,
   $\gamma < \angle C \rightarrow \cos\angle B\cos\alpha + \cos\angle A\cos\beta > 0$;

6. $(D > 0 \wedge \alpha < \angle A) \rightarrow (\beta < \angle B \vee \gamma < \angle C)$,
   $(D > 0 \wedge \beta < \angle B) \rightarrow (\gamma < \angle C \vee \alpha < \angle A)$,
   $(D > 0 \wedge \gamma < \angle C) \rightarrow (\alpha < \angle A \vee \beta < \angle B)$;

7. $D > 0 \rightarrow [\,\alpha < \angle A \vee \alpha > \pi - \angle A \vee \beta < \angle B \vee \beta > \pi - \angle B \vee \gamma < \angle C \vee \gamma > \pi - \angle C\,]$.

The first three items in this list are rather easily proved. The remaining four items require a significantly more technical analysis. Also, experimental evidence suggests that this collection of constraints is necessary and sufficient for a suitable tetrahedron ABCP to exist for a given acute triangle ABC. At present though, the sufficiency claim is only a conjecture. (Note: the arrows mean logical implication here.)