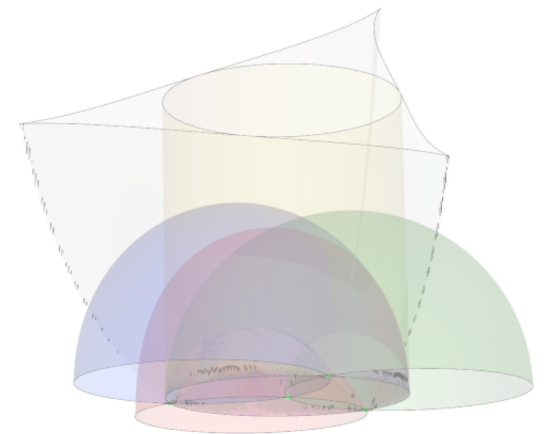
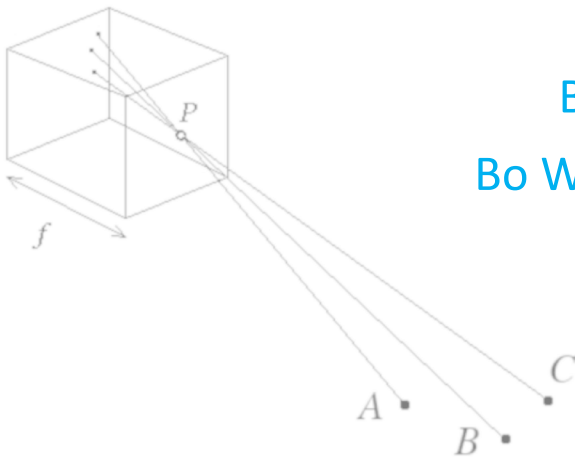


Blowing up Control Points

Algebraic Geometry Applied to the
Perspective Three-Point Problem

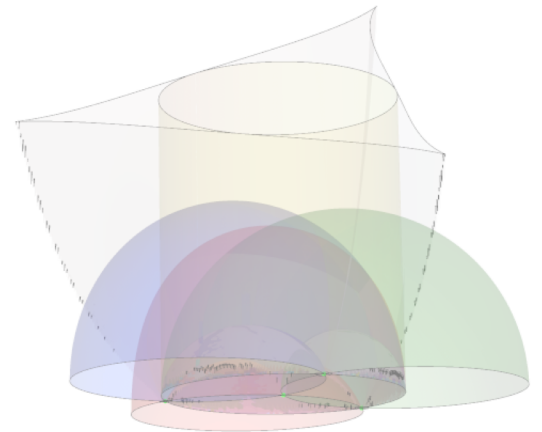
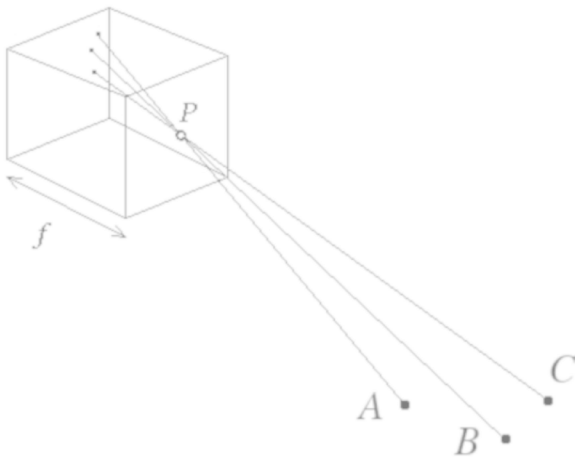
Michael Q. Rieck, Drake University

Based in part on discussions with
Bo Wang (王波), Institute of Automation,
Chinese Academy of Sciences



Contents

1. Strongly and Weakly Related Points
2. Toroids
3. Applying Basic Algebraic Geometry
4. Counting and Finding Related Points
5. More Algebraic Geometry
6. Closing Questions



Strongly and Weakly Related Points

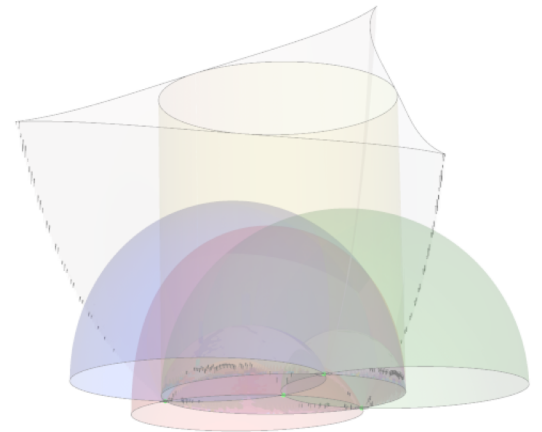
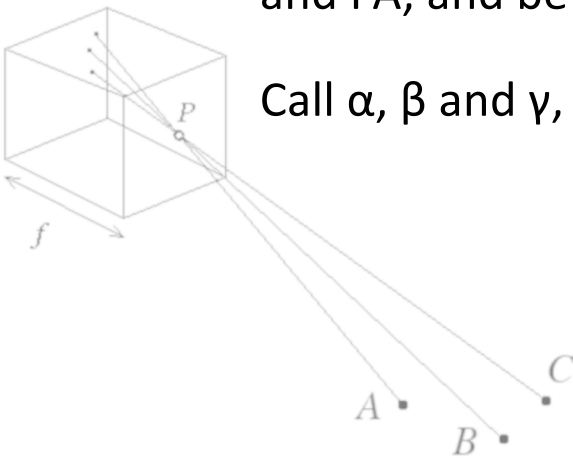
Only have time for a fast introduction to the P3P Problem

Fix three points A, B, C in three-dimensional real space (not collinear), and call these the *control points*

Let P denote an arbitrary point, other than A, B or C

Let α, β and γ be the angles between rays PB and PC , between rays PC and PA , and between rays PA and PB , respectively

Call α, β and γ , the *view angles*



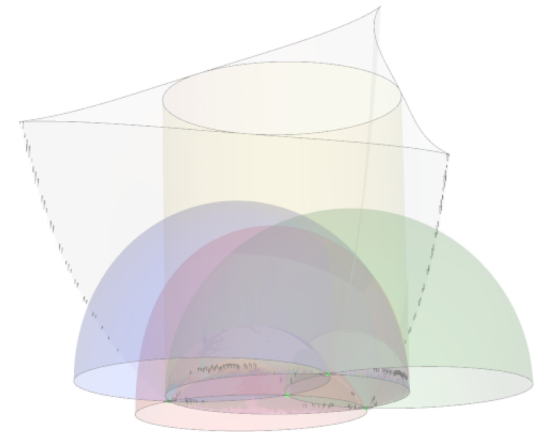
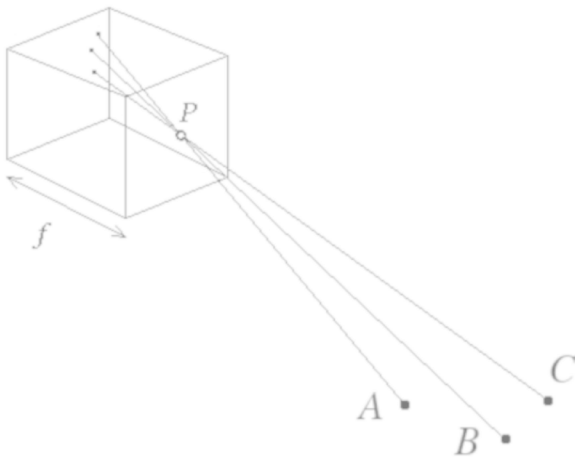
Strongly and Weakly Related Points

Let $c_1 = \cos \alpha$, $c_2 = \cos \beta$ and $c_3 = \cos \gamma$

Let r_1 , r_2 and r_3 be the distance from P to the control points A , B and C , respectively (the *unknowns* in the P3P Problem)

Let d_1 be the distance between B and C , let d_2 be the distance between C and A , and let d_3 be the distance between A and B

Regard d_1 , d_2 and d_3 as the side lengths of the *control points triangle* ABC

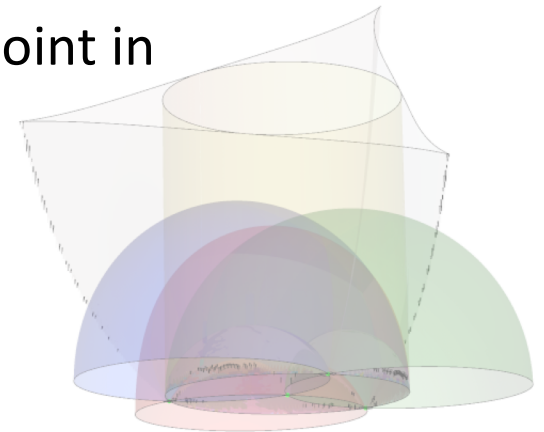
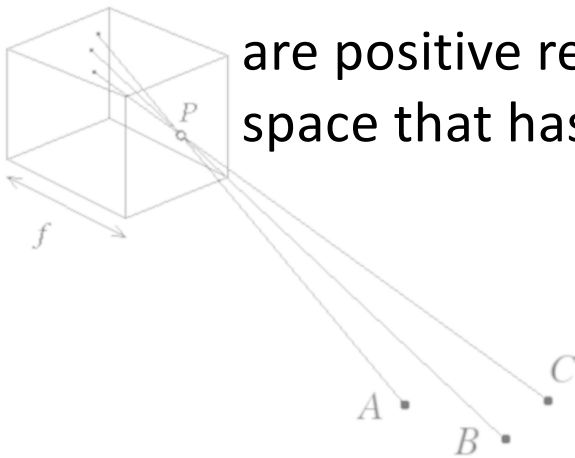


Strongly and Weakly Related Points

Using the Law of Cosines, we obtain *Grunert's system* of equations, where r_1 , r_2 and r_3 are the unknowns in the P3P Problem:

$$\begin{cases} r_2^2 + r_3^2 - 2 c_1 r_2 r_3 = d_1^2 \\ r_3^2 + r_1^2 - 2 c_2 r_3 r_1 = d_2^2 \\ r_1^2 + r_2^2 - 2 c_3 r_1 r_2 = d_3^2 \end{cases}$$

When this is solved (multiple solutions), each solution for which the r 's are positive real numbers, corresponds to a point in space that has the correct view angles

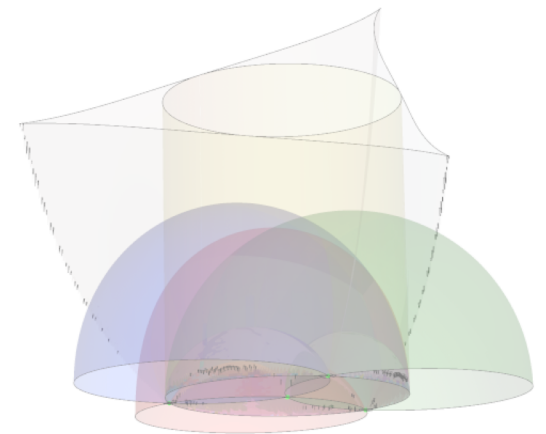
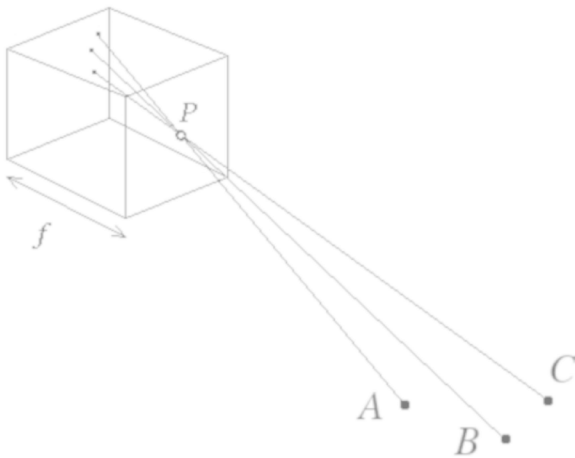


Strongly and Weakly Related Points

Use a Cartesian coordinate system (respecting the physical angles), with coordinates x , y and z , is used so that the control points all lie on the unit circle in the xy -plane

Letting $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$ be the Cartesian coordinates of A , B and C , respectively, it is further assumed that there are angles ϕ_1 , ϕ_2 and ϕ_3 for which $x_\alpha = \cos \phi_\alpha$ and $y_\alpha = \sin \phi_\alpha$, for $\alpha = 1, 2, 3$, and such that $\phi_1 + \phi_2 + \phi_3 = 0$

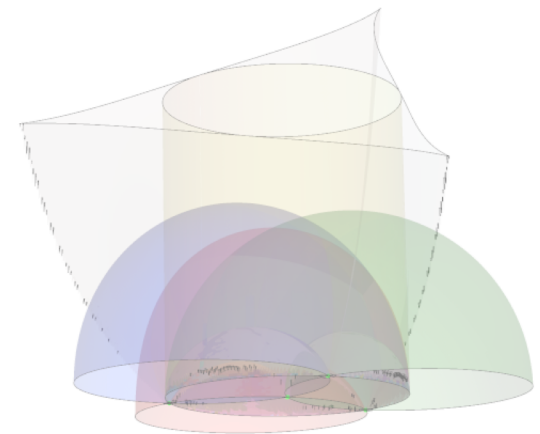
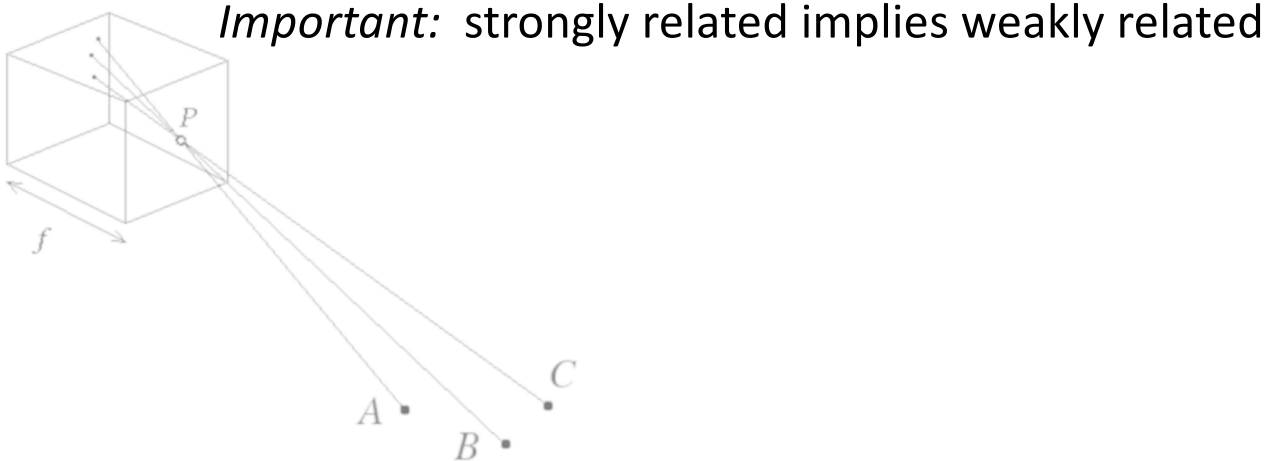
Any reasonable coordinate system can be easily converted to a coordinate system of this sort



Strongly and Weakly Related Points

Two non-control points in xyz-space will be called *strongly related* if their Cartesian coordinates yield the same values for the c_1 , c_2 and c_3

Two non-control points in xyz-space will be called *weakly related* if their Cartesian coordinates yield the same values for the quantities c_1^2 , c_2^2 , c_3^2 and $c_1 c_2 c_3$; in other words, they have the same values for $|c_1|$, $|c_2|$ and $|c_3|$, and disagree in an even number (0 or 2) of the signs of the cosines

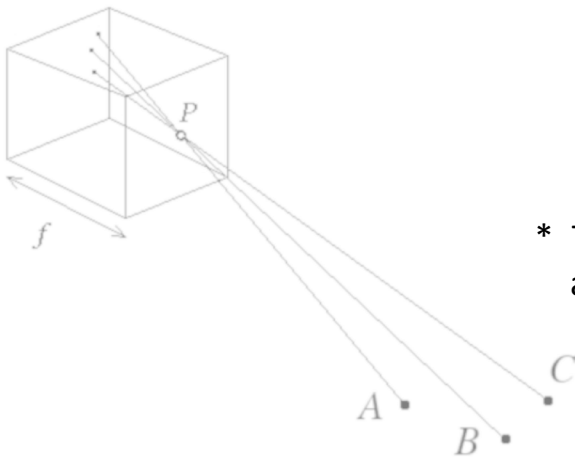


Strongly and Weakly Related Points

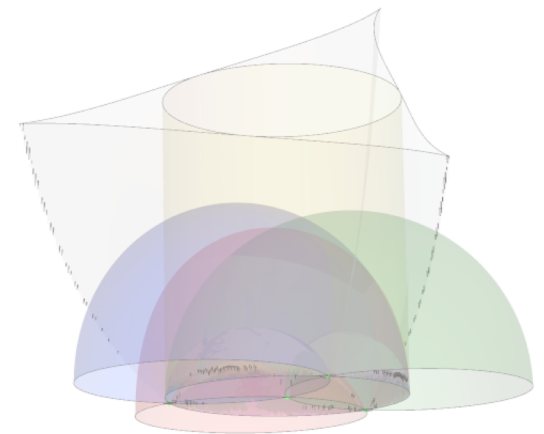
Consider two non-control points P and Q

P and Q are strongly related if and only if the configuration of three rays from P to a control points can be rigidly moved* to the corresponding configuration of rays from Q

P and Q are weakly related if and only if the configuration of the three lines through P and a control points can be rigidly moved* to the corresponding configuration of lines through Q



* Translated and/or rotated, but I should also allow for a possible reflection here

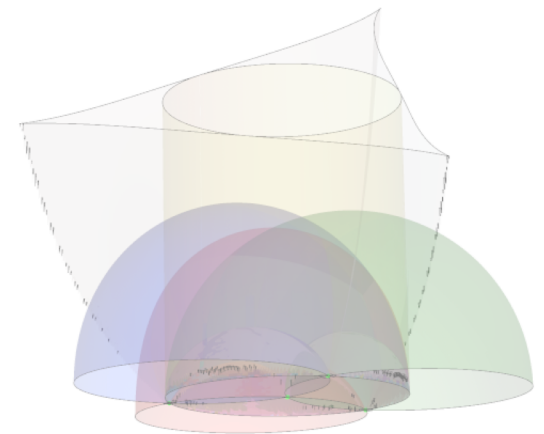
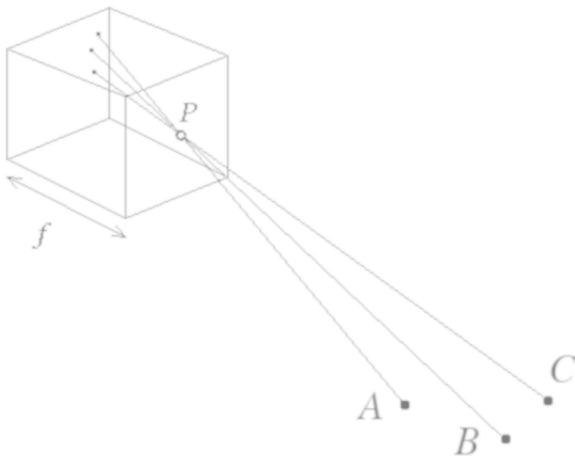


Strongly and Weakly Related Points

For given values of c_1 , c_2 and c_3 , there can be up to *eight* points in real xyz-space with these cosine values (they are strongly related; they have the same view angles)

However, two points that are mirror images of each other, across the xy-plane, are strongly related, so in a sense, “eight” is double counting

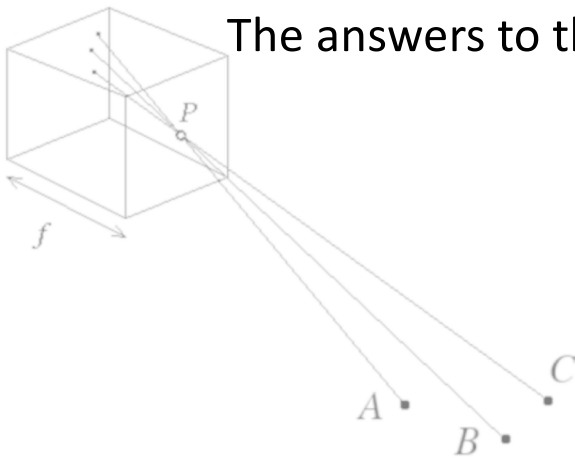
Often, it is desirable to focus only on the points in the upper half-space ($z > 0$), and ignore their reflections in the lower half-space ($z < 0$)



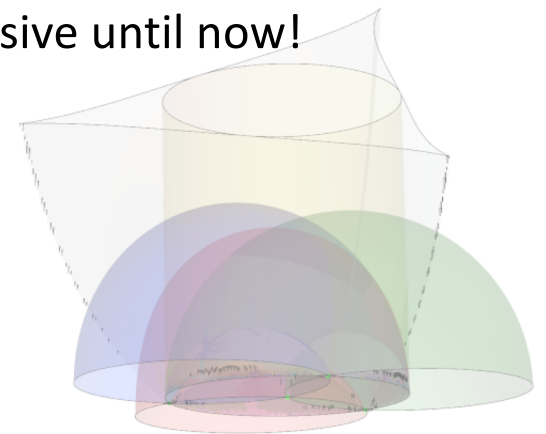
Strongly and Weakly Related Points

Here are a few interesting and practical questions related to P3P:

1. Given the parameters d_1 , d_2 , d_3 , c_1 , c_2 and c_3 , how many corresponding points in three-dimensional real space (xyz-space) are there?
2. Given a non-control point in xyz-space, how many strongly related points does it have?
3. Where are they located?



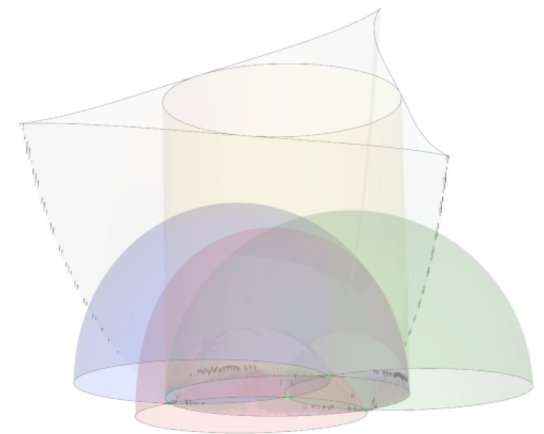
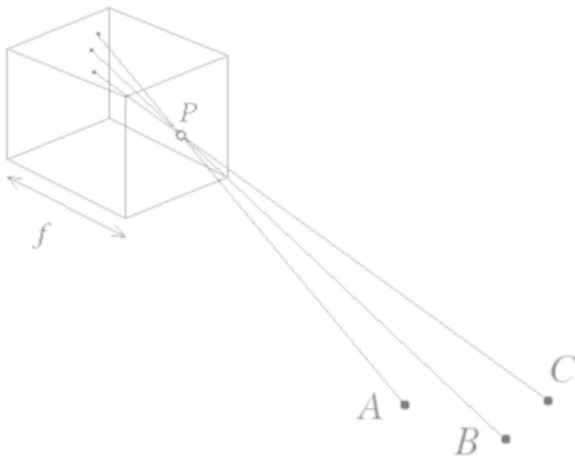
The answers to these questions have been mostly elusive until now!



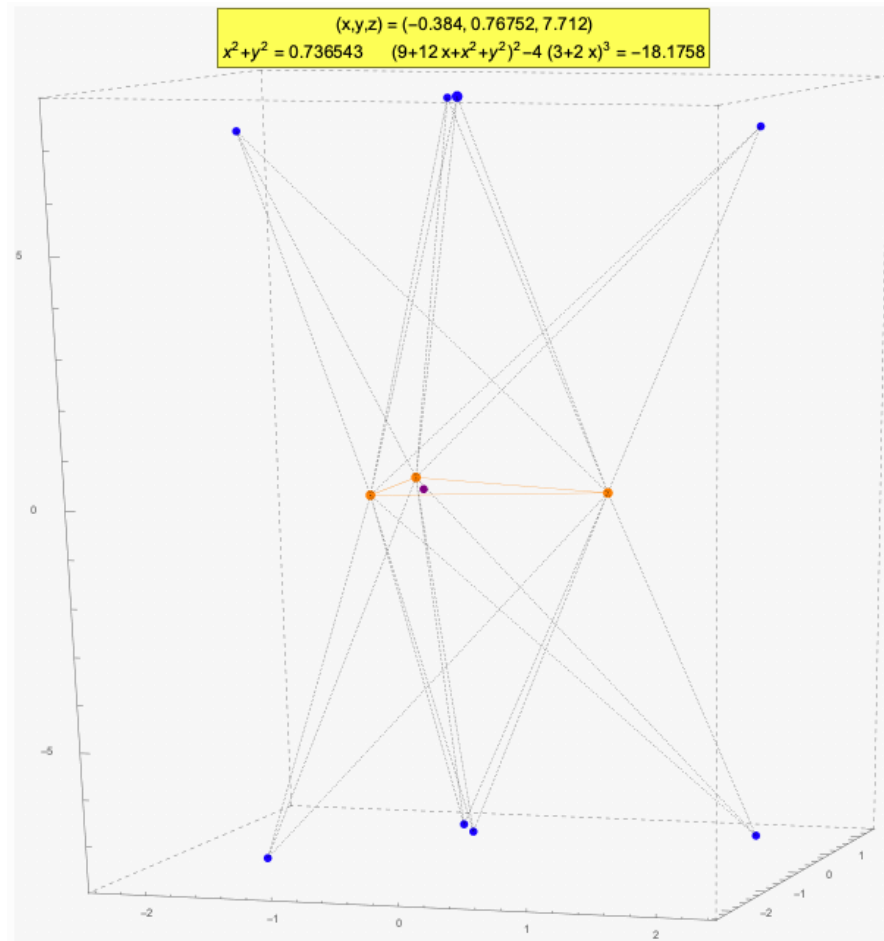
Strongly and Weakly Related Points

In the past, there has been no interest in, or even discussion of, weakly related points, but as we will now see, the notion of “weakly related” exhibits *better mathematical behavior* than “strongly related,” from the standpoint of *continuity*

It will be helpful to consider *particles* (i.e. “moving points”), regarding a camera’s optical center and its weakly related points (including its strongly related points) as such



Strongly and Weakly Related Points



Related Particles

The orange dots are the control points

The purple dot is the *orthocenter* of the control points triangle

All of the blue dots are strongly related to each other

One of these, the biggest blue dot, is presumed to be a camera's position, whose coordinates can be seen at the top

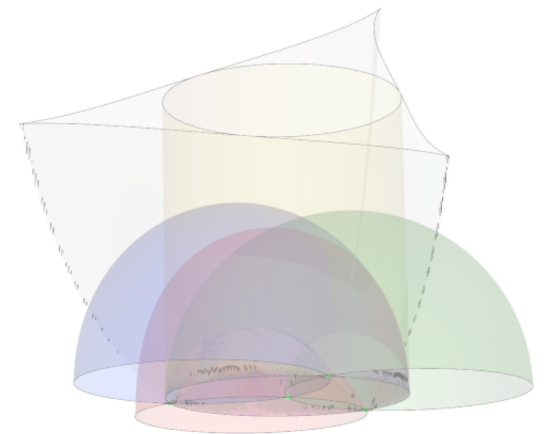
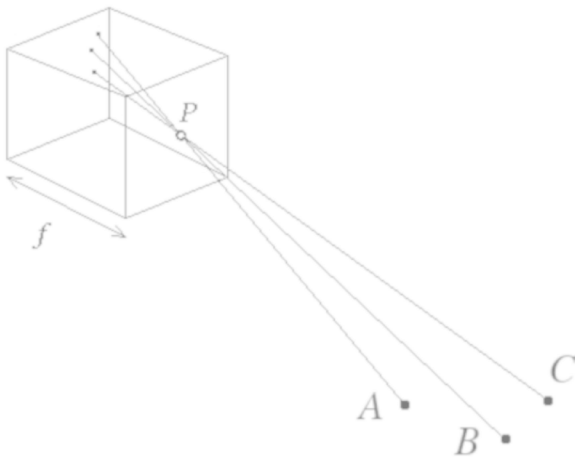
The green dots are only weakly related to the blue dots

Strongly and Weakly Related Points

As the camera's optical center moves continuously around in xyz-space, its weak relatives (including its strong relatives) move around mostly without being impeded, though they *often* pass through control points

In passing through a control point, such a particle can switch between being strongly related and being merely weakly related to the continuously moving optical center

Also, at certain locations (to be discussed later) weakly related particles can come together and then disappear (due to complex solutions)

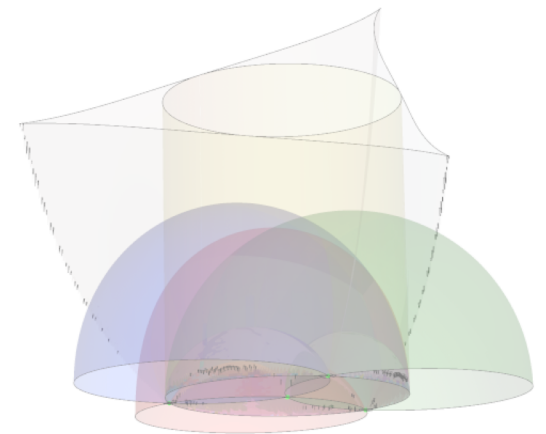
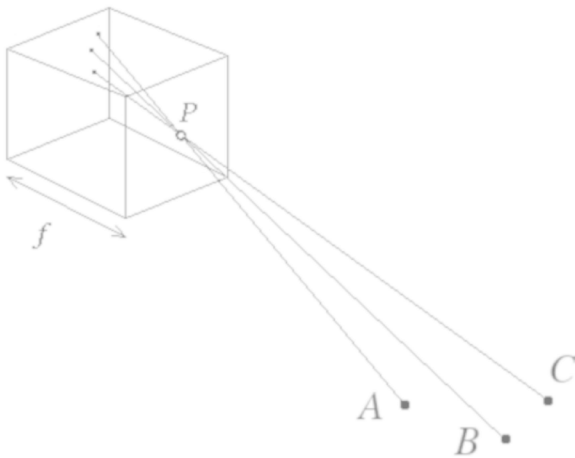


Toroids

The word “toroid” has multiple definitions, but here is the one that is useful for the P3P problem:

A *toroid* is the surface that results in three-dimensional (real) space, when an arc of a circle is rotated about the line through its endpoints

This surface is smooth, except at the two endpoints, where the surface is *singular*



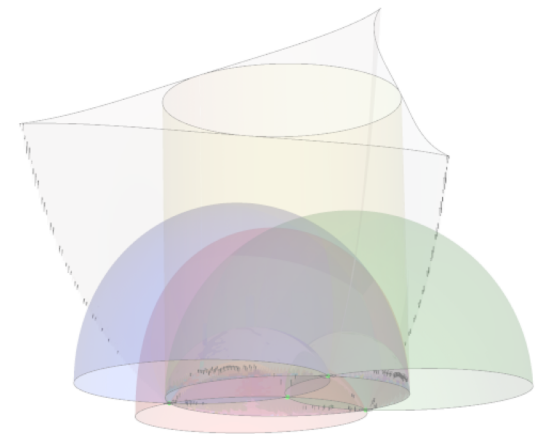
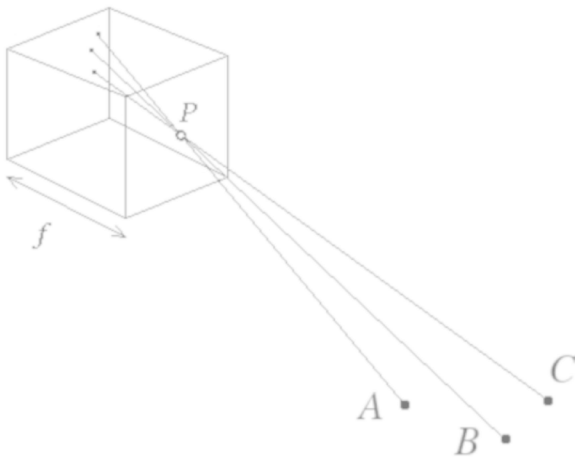
Toroids

The *circumcircle* is the circle containing the three triangle vertices A, B and C

Let us now define the three *basic toroids*, as follow:

Toroid_A is generated from the circular arc through A whose endpoints are B and C , and similarly for *Toroid_B* and *Toroid_C*

We also need another toroid, *Toroid _{$\pi-A$}* , which is generated from the other arc on the circumcircle, connecting B and C, the arc that does *not* include A



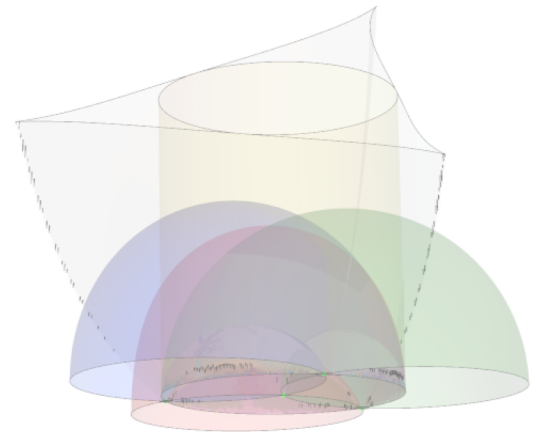
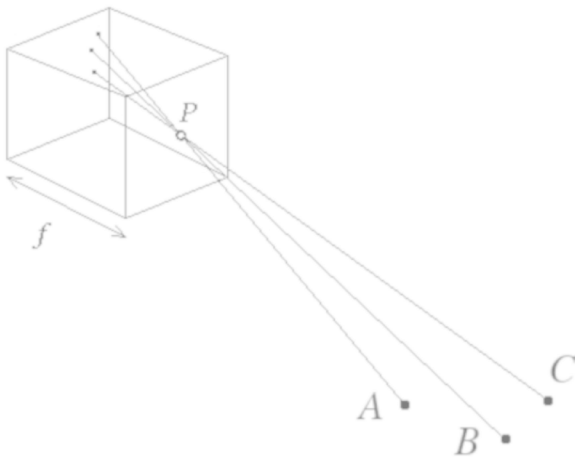
Toroids

Similarly for $Toroid_{\pi-B}$ and $Toroid_{\pi-C}$

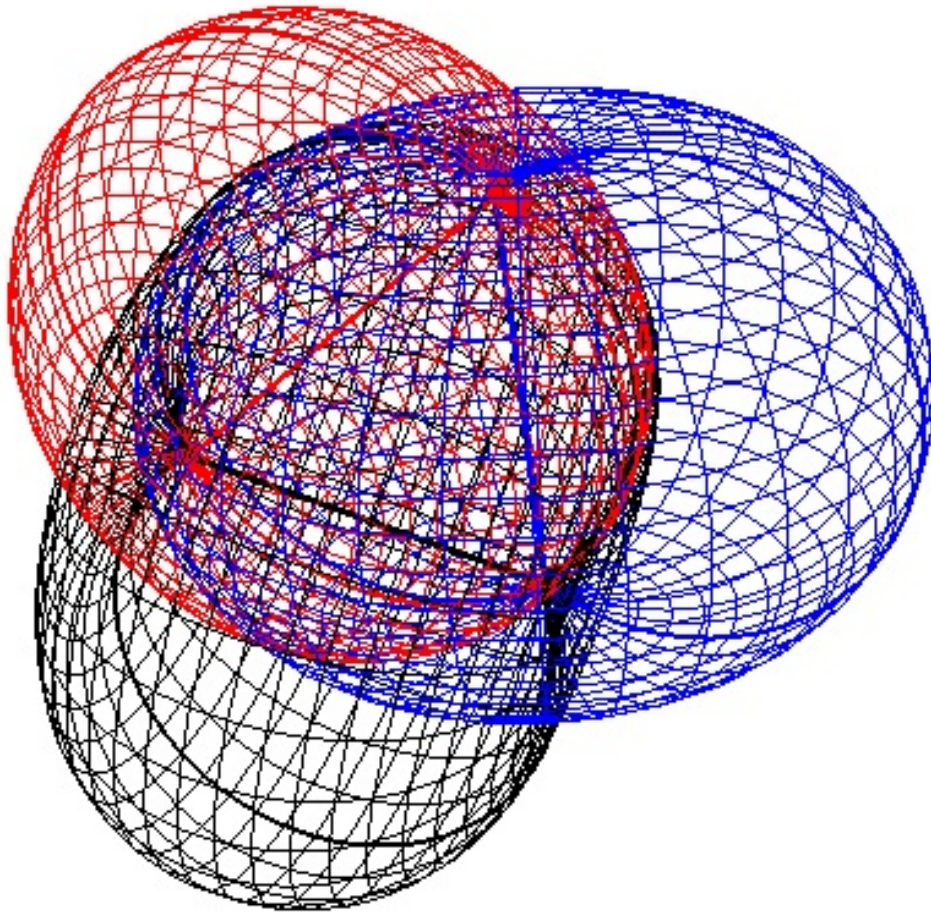
The union of $Toroid_A$ and $Toroid_{\pi-A}$ will be called a *basic double toroid*, and denoted $DToroid_A$

It is the surface you get when you rotate the entire circumcircle about the line through B and C

Similarly for $DToroid_B$ and $DToroid_C$



Toroids



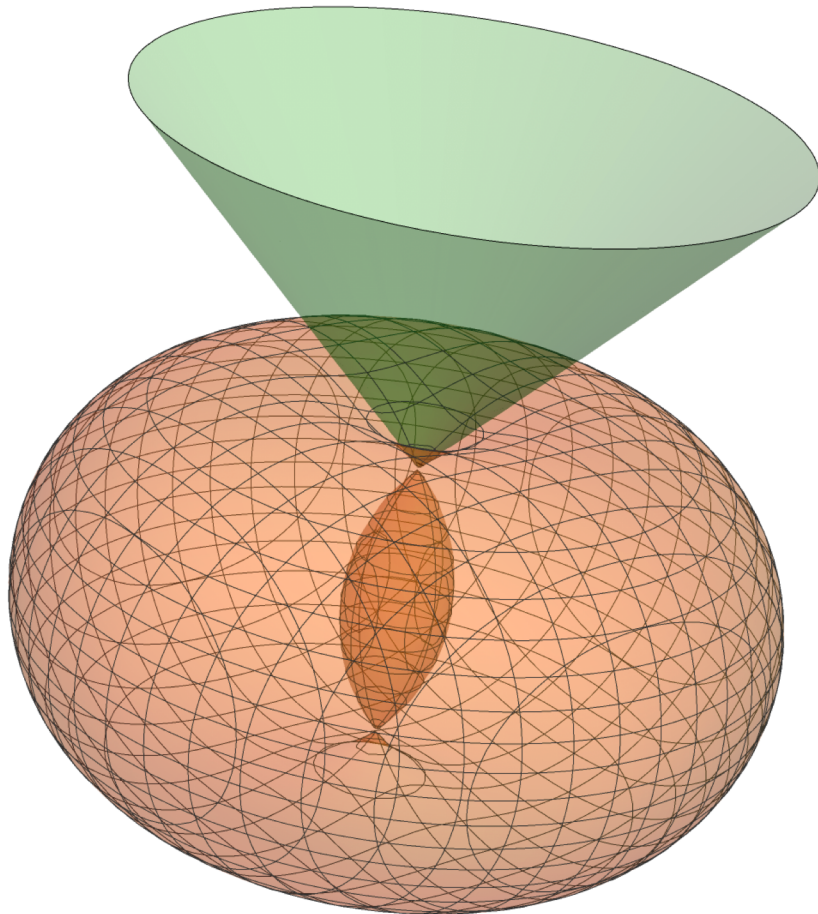
The three basic toroids:

Toroid_A*, *Toroid_B* and *Toroid_C

When the camera's optical center is outside of the union of *Toroid_A*, *Toroid_B* and *Toroid_C*, all of its weak relatives are outside too, and they are strongly related.

(image from Wang, B., Hu, H. and Zhang, C.)

Toroids



A double toroid, and the tangent cone for its outer toroid at one of the singularities

Notice the inner toroid inside the outer toroid

The cone's apex is of course one of the two singular points of the double toroid

Each ray on the cone, from its apex, is tangent to the outer toroid

The tangent cones allow us to determine several things about the toroids and their intersections

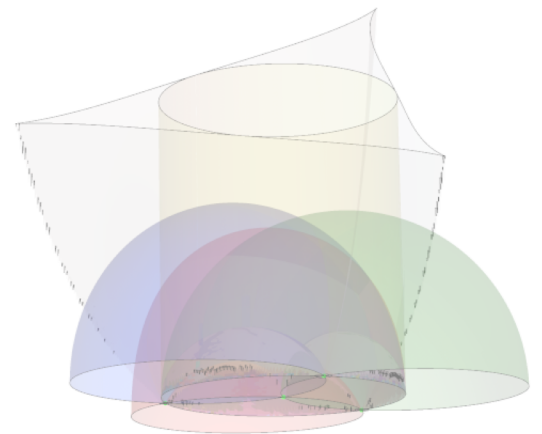
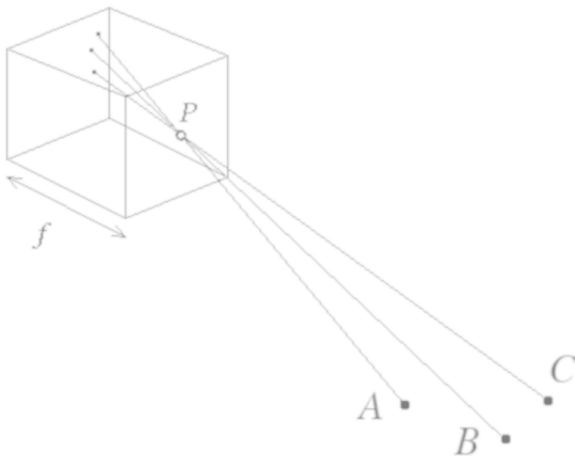
Cute fact: The angle, at any point on the outer toroid, between the rays from it to the toroid's two singularities, is equal to the angle, at the cone's apex, between the cone's half-axis and any of its rays from the apex

Toroids

If the triangle ABC 's interior angle at C is acute, then $Toroid_C$ uses the longer of the two circular arcs connecting A and B , forming an *outer toroid* that is outside the unit sphere

If instead the triangle ABC 's interior angle at C is obtuse, then $Toroid_C$ uses the shorter of the two circular arcs connecting A and B , forming an *inner toroid* that is inside the unit sphere

Similarly for $Toroid_A$ and $Toroid_B$

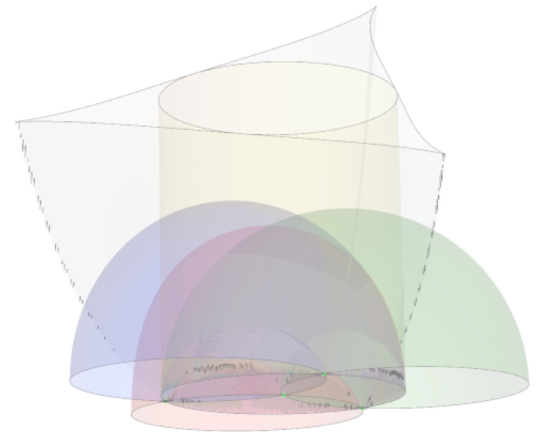
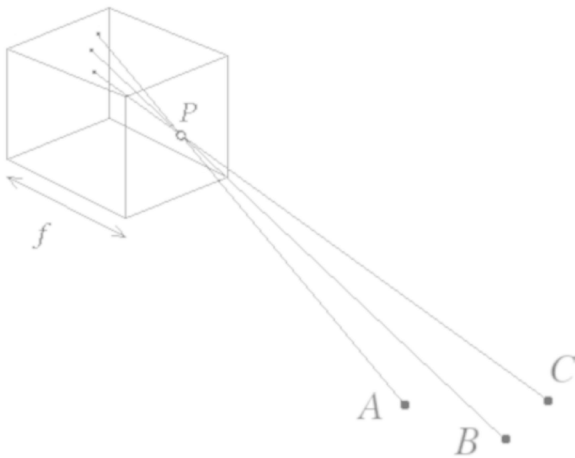


Toroids

Here is another crucial fact for analyzing related points:

If a smoothly moving particle travels through one of the three basic double toroids (in an ordinary way), then *exactly one* of its weak relatives in the upper half-space (and its reflection in the lower half-space) will pass through the corresponding control point

Moreover, whenever a particle travels through a control point, *all* of its weak relatives will pass through the corresponding basic double toroid

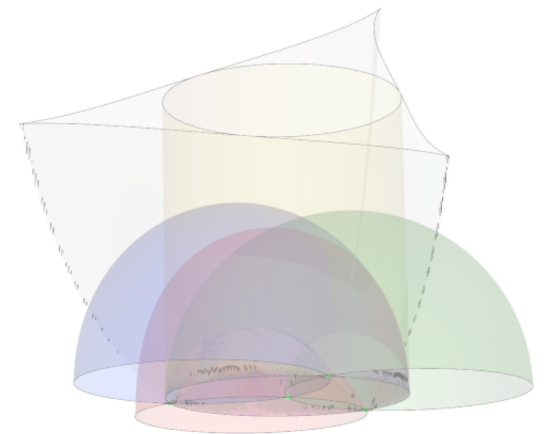
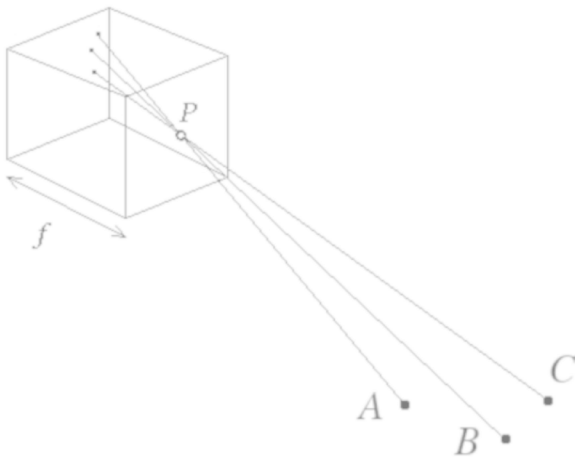


Toroids

Henceforth, assume that the control points triangle ABC is **acute**, so $A < \pi/2$, $B < \pi/2$ and $C < \pi/2$ (where A, B, C mean the interior angles here)

Three-dimensional real space (*i.e.* xyz-space) will be divided into regions, based on being outside $Toroid_A$, versus being inside $Toroid_{\pi-A}$, versus being between $Toroid_{\pi-A}$ and $Toroid_A$, and similarly using B and C in place of A

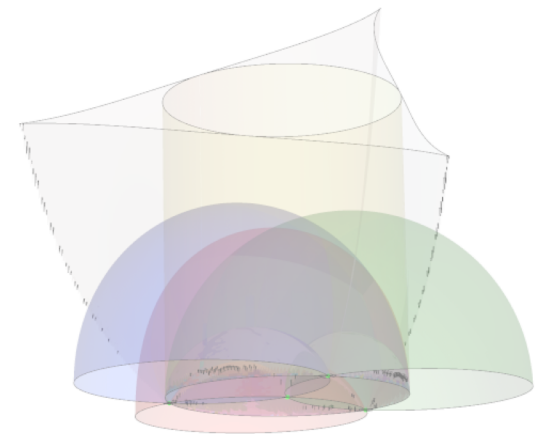
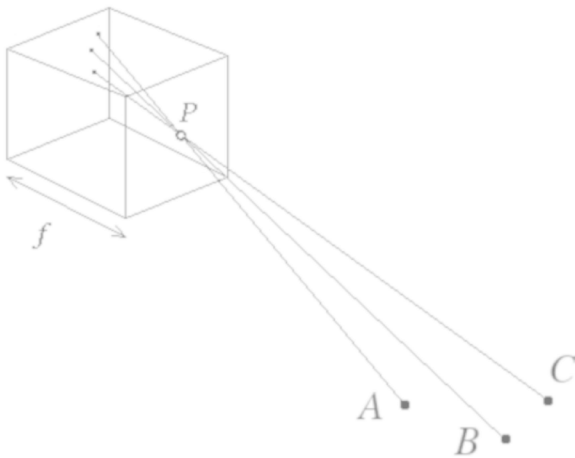
In this way, we will speak of *toroidal regions*



Toroids

Specifically, we will speak of toroidal region abc , where each of a , b and c is either 0, 1 or 2, as follows:

- Each digit concerns dividing space based on one of the basic double toroids, with a referring to $DToroid_A$, b referring to $DToroid_B$, c referring to $DToroid_C$
- If $a = 0$, then the region is outside $Toroid_A$; if $a = 1$, then the region is between $Toroid_{\pi-A}$ and $Toroid_A$; if $a = 2$, then the region is inside $Toroid_{\pi-A}$
- Similarly for the digits b and c

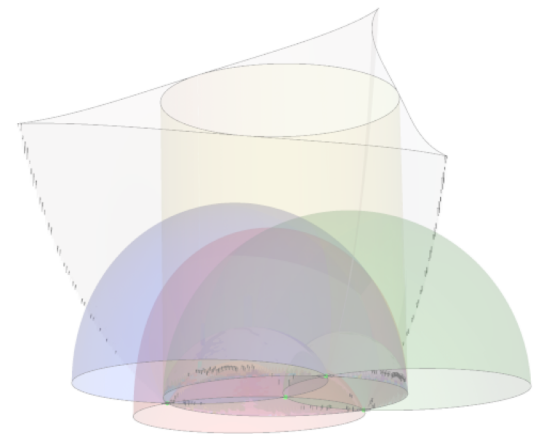
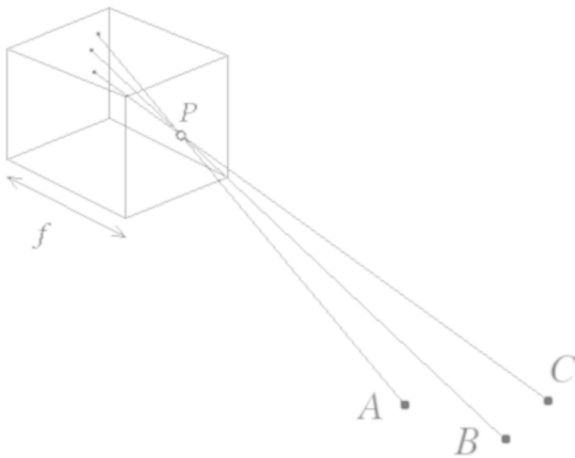


Toroids

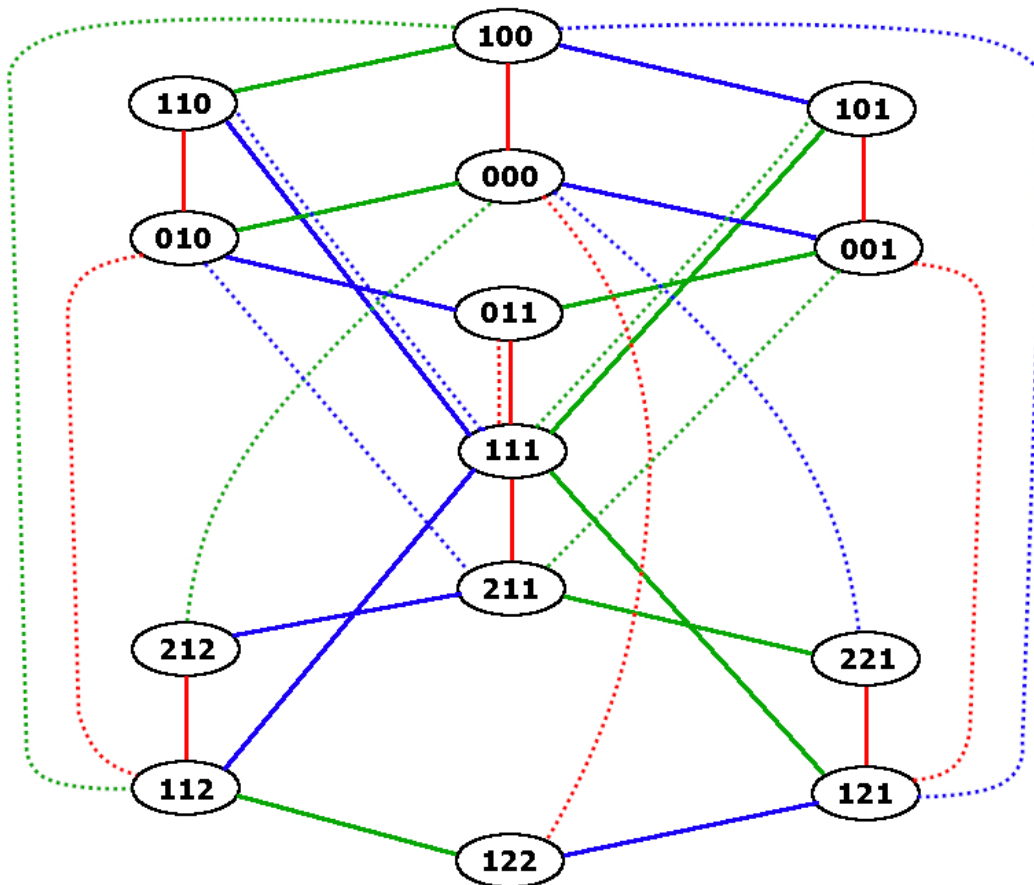
For instance, toroidal region 010 is outside both $Toroid_A$ and $Toroid_C$, but between $Toroid_{\pi-B}$ and $Toroid_B$

Observe that a toroidal region *descriptor abc* can never include a 0 and a 2, because a 0 implies being outside the unit sphere, while a 2 implies being inside the unit sphere

Also, 222 is not valid since $Toroid_{\pi-A}$, $Toroid_{\pi-B}$ and $Toroid_{\pi-C}$ intersect in a single point, *i.e.* the orthocenter, and there are no points in the interiors of all three of these toroids



Toroids



The Toroidal Region Transition Graph

This graph describes all the ways that a smoothly moving particle can move between toroidal regions

A solid edge represents a transition caused when a particle crosses a basic double toroid in an ordinary way (red for $DToroid_A$, green for $DToroid_B$, and blue for $DToroid_C$)

A dotted edge represents a transition caused by passing through a control point (red for A, green for B, and blue for C)

The details are pretty straightforward, and explained in a recently submitted paper

This graph is crucial to locating strong and weak relatives

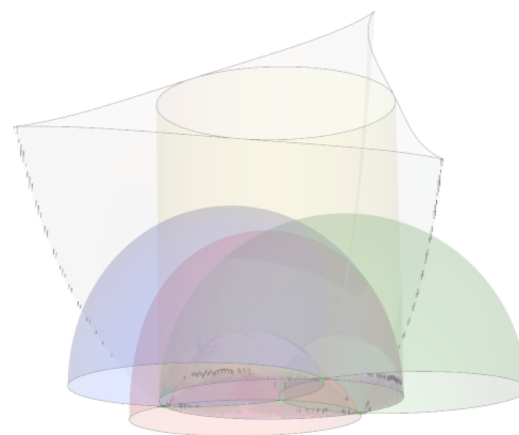
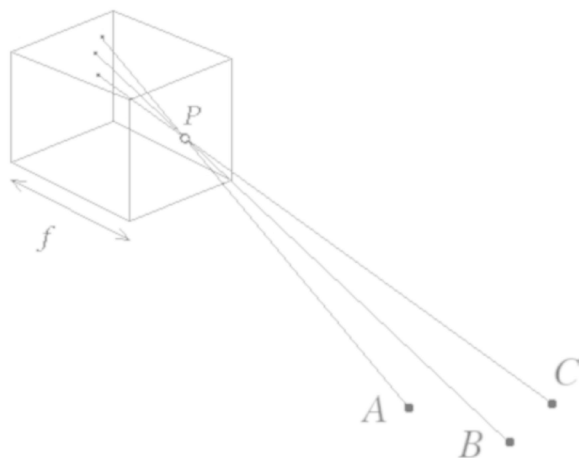
Applying Basic Algebraic Geometry

Algebraic geometry is primarily focused on studying systems of polynomial equations in several variables, and the sets of points in space that satisfy such systems

We will here oversimplify this very important area of mathematics

The set of points is called a *variety*, but we may also refer to it as a “space”

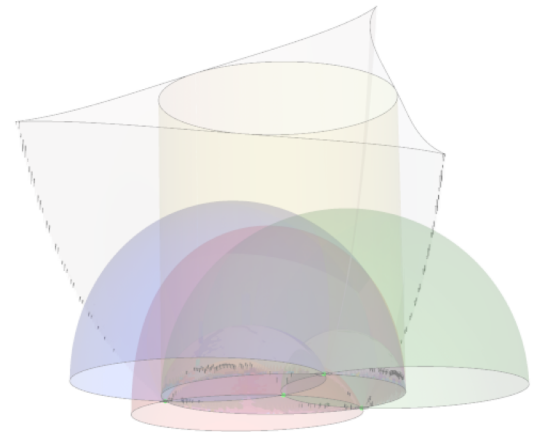
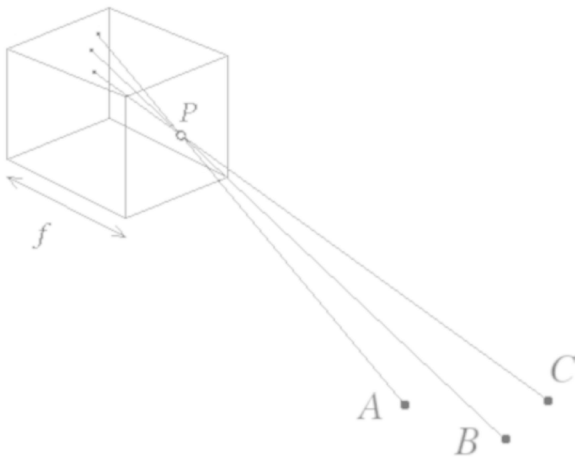
A variety has an associated dimension, which is rather intuitive



Applying Basic Algebraic Geometry

As an example, for fixed parameters $d_1, d_2, d_3, c_1, c_2, c_3$, consider Grunert's system of equations in the variables (unknowns) r_1, r_2 and r_3

The solutions are a finite number of isolated points in the 3D space of all possible triples (r_1, r_2, r_3) , and so this variety has dimension zero

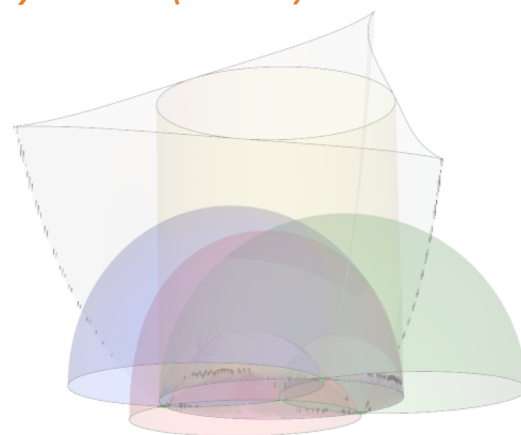
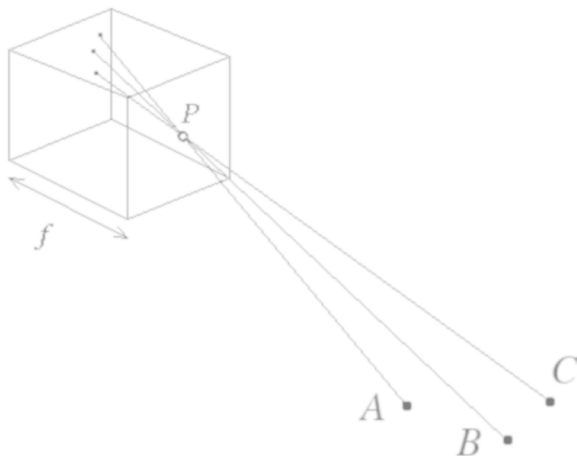


Applying Basic Algebraic Geometry

As another example, consider all of the points in xyz-space for which $(9 + 12 x_L + x_L^2 + y_L^2)^2 = 4 (3 + 2 x_L)^3$, where x_L and y_L are given by certain rational functions of x , y and z (that time does not allow me to explain)

After “clearing out denominators,” we get a single polynomial equation in x , y and z

The variety for this is the union of two surfaces, one called the *danger cylinder* and the other called the *companion surface to the danger cylinder (CSDC)*



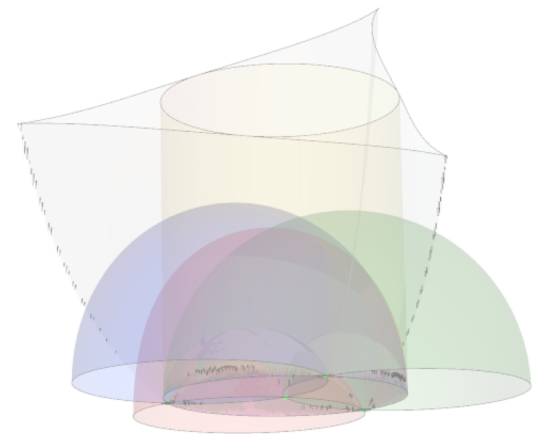
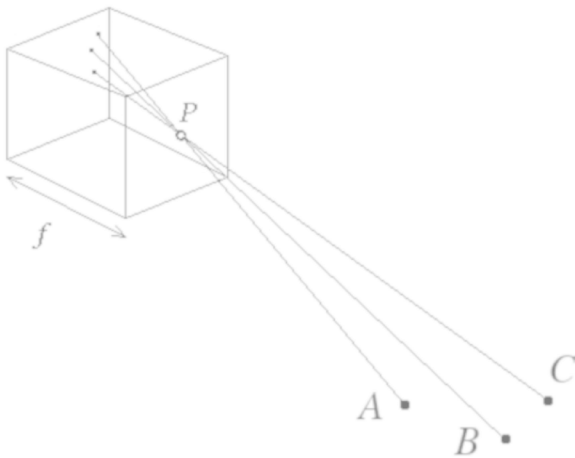
Applying Basic Algebraic Geometry

The danger cylinder is just the circular cylinder that contains the circumcircle

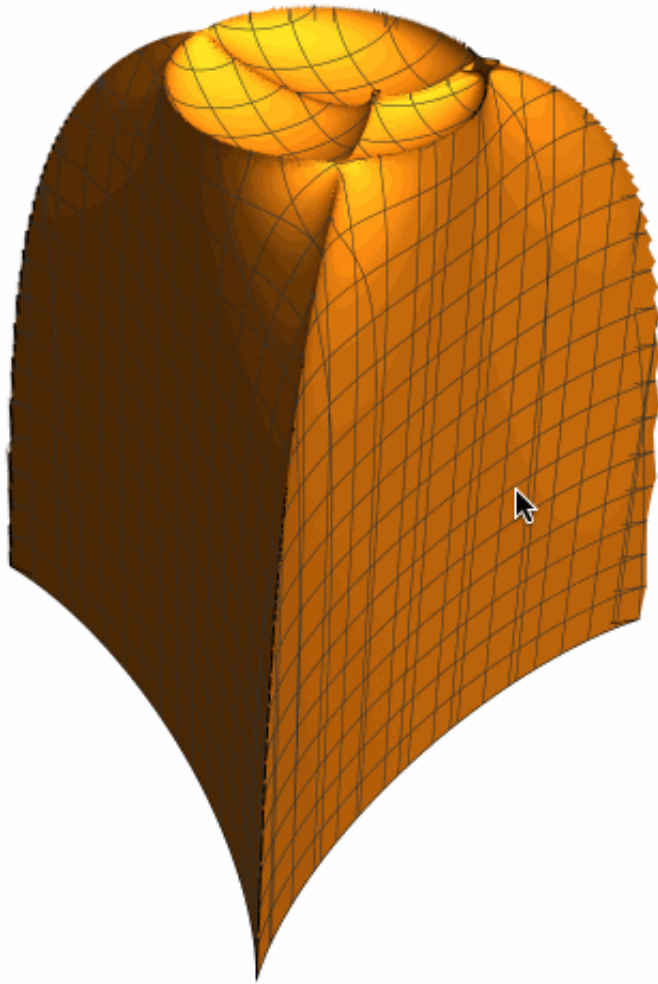
It has long been recognized that any point on the danger cylinder corresponds to a repeated solution of Grunert's system (using its cosine values for parameters)

So, if a particle travels onto the danger cylinder, then it will collide with a weak relative

Moreover, if a particle travels onto the *CSDC*, then two of its weak relatives will collide on the danger cylinder



Applying Basic Algebraic Geometry



Half of the *CSDC*, in the case where the triangle ABC is acute, and so its orthocenter is inside its circumcircle

The other half is the reflection of this about the xy -plane

The vertices (A, B, C) can be seen here, as can the circumcircle, and inside this, the orthocenter H

(Ignore the gap near a vertex, which is a rendering issue)

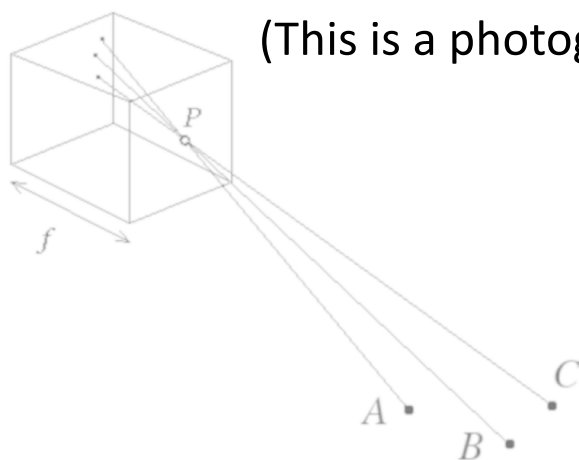
Applying Basic Algebraic Geometry

Our basic double toroids are also examples of varieties of dimension two

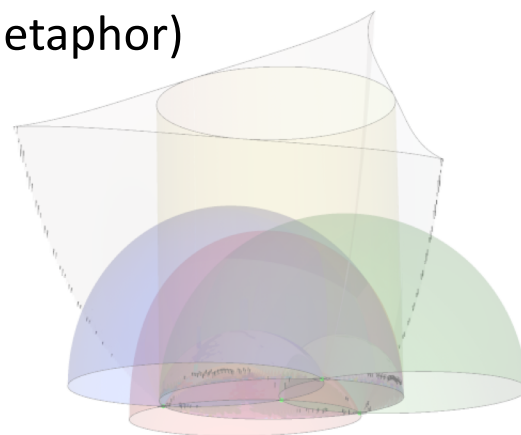
Varieties often have non-smooth points called *singularities*

Each double toroid has two such singularities

Blowing up points is one standard way to tame singularities in algebraic geometry



(This is a photography metaphor, not a demolition metaphor)

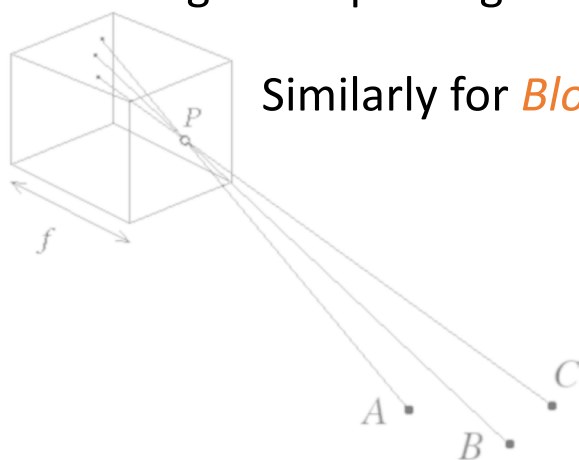


Applying Basic Algebraic Geometry

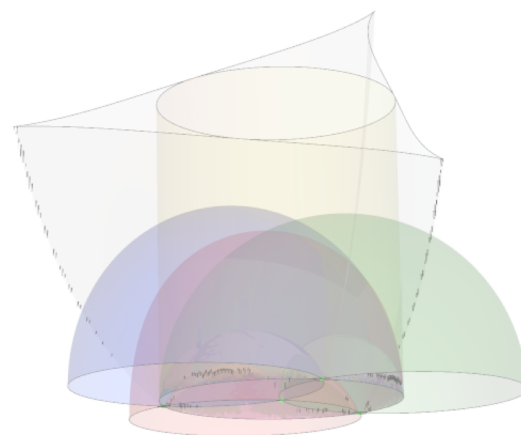
The *blowup* of a point, say A , which we will denote Blowup_A , is technically just the set of all lines through A

Since we are working here in three-dimensional real space, this set of lines forms a *projective plane*

We want to think of the blowup of A , as representing tangent lines to smooth curves that pass through A , and so each such line describe the direction we are using when passing through the point A



Similarly for Blowup_B and Blowup_C , of course

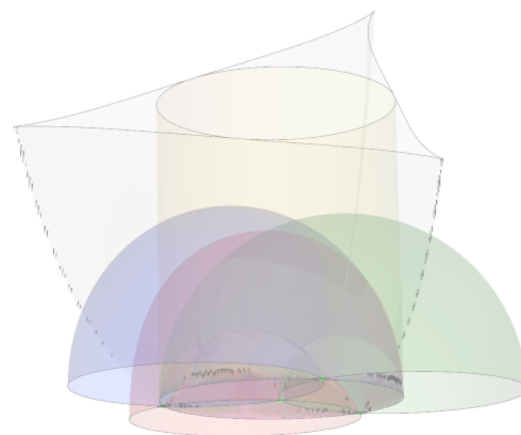
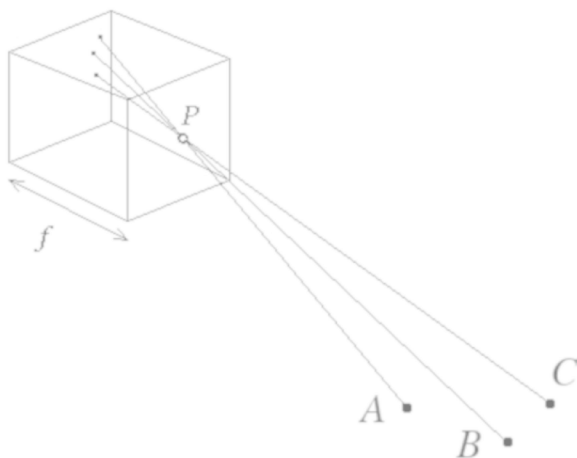


Applying Basic Algebraic Geometry

We will actually also need a closely related and equally simple concept, as follows

Let $D\text{blowup}_A$ denote the set of all rays that originate at the point A , thinking of such a ray as indicating a direction *from* which A is being approached along a smooth curve

It is natural to model $D\text{blowup}_A$ as a tiny sphere around A , identifying each point on the sphere with the ray that passes through it

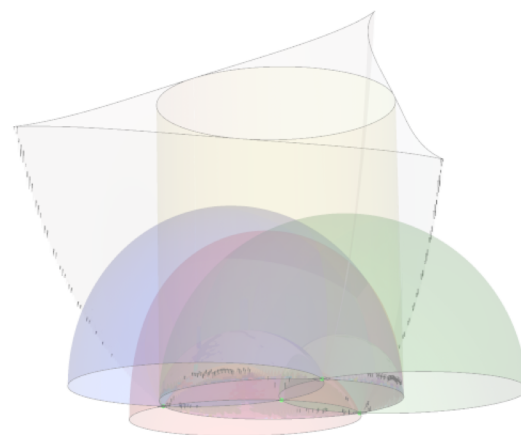
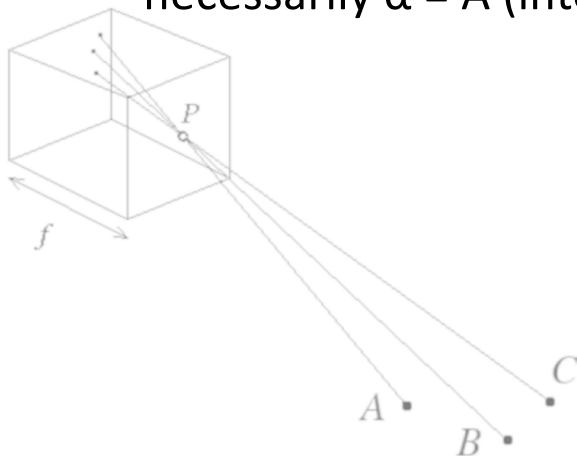


Applying Basic Algebraic Geometry

We will call $D\text{Blowup}_A$ the *double cover* of Blowup_A due to the fact that elements of $D\text{Blowup}_A$ naturally correspond to elements of Blowup_A in a two-to-one manner (two rays per line)

Think of an element of $D\text{Blowup}_A$ as describing a point that is very close to A (infinitesimally close, really!)

It makes sense to speak about the view angles α , β and γ for such a point, but necessarily $\alpha = A$ (interior angle at A)



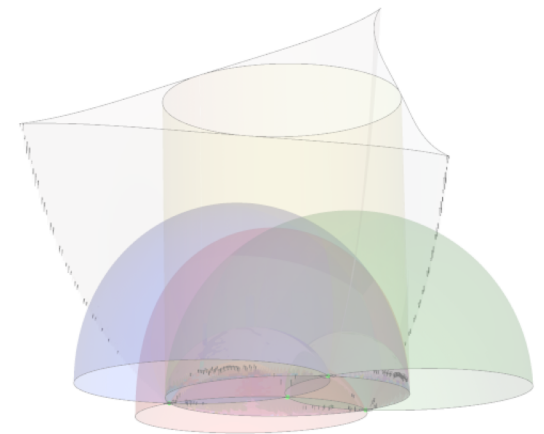
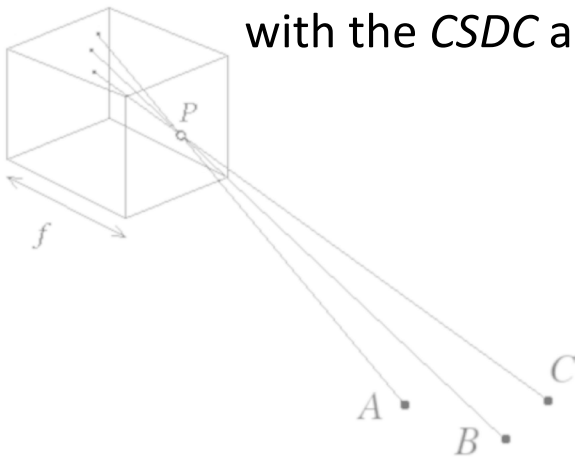
Applying Basic Algebraic Geometry

Treating the points (elements) of $DBlowup_A$ as limiting points for ordinary points, and extending the quantities α , β and γ to these has proven to be very useful

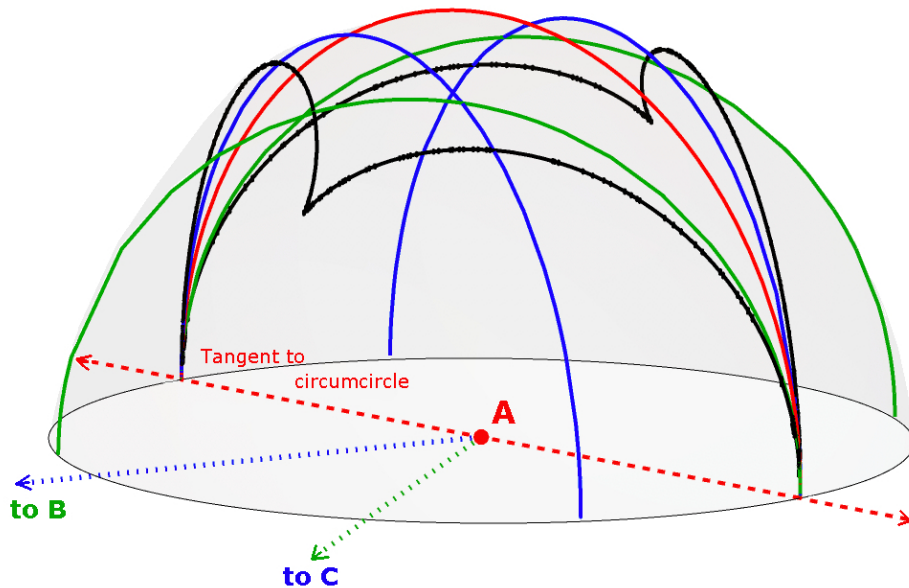
The notions of weakly related and strongly related points are naturally extended to these limit points too

Obviously the situation is similar for $DBlowup_B$ and $DBlowup_C$

In the next two slides, the intersection of $DBlowup_A$ with the basic toroids and with the $CSDC$ are revealed



Applying Basic Algebraic Geometry



Intersecting $DBlowup_A$ with other Surfaces

Only the top half of $Blowup_A$ is seen here

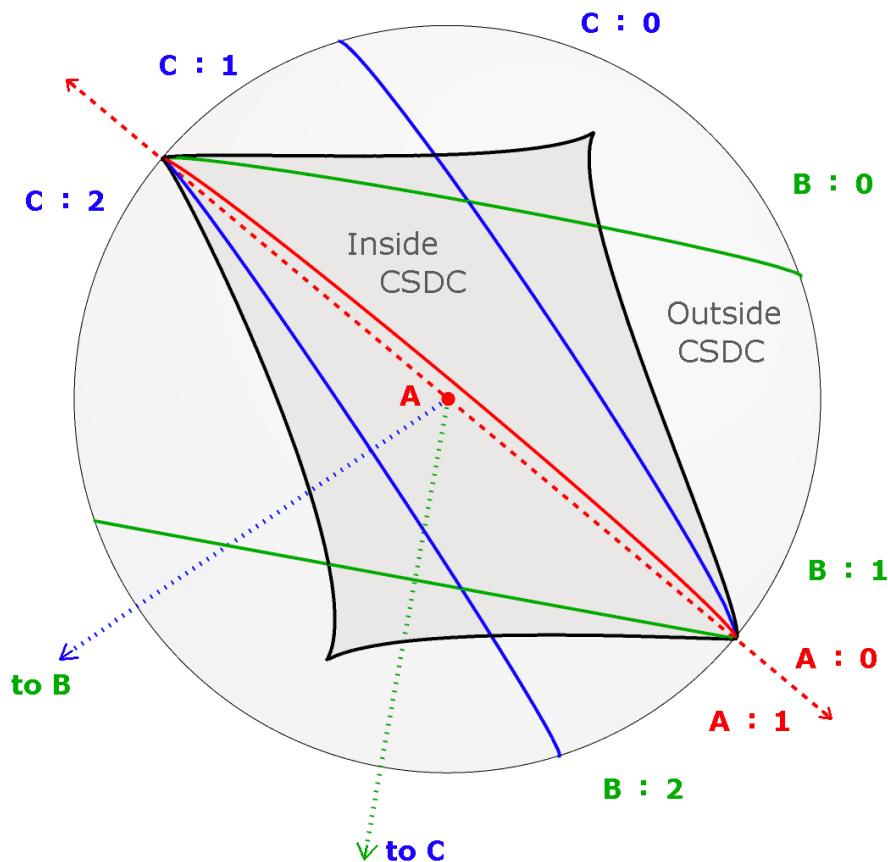
$Toroid_A$ cuts $DBlowup_A$ into two halves via a great circle (red), inside and outside $Toroid_A$

$DToroid_B$ cuts $DBlowup_A$ into three parts: outside $Toroid_B$, inside $Toroid_{\pi-B}$, and between $Toroid_{\pi-B}$ and $Toroid_B$ (green)

Similarly for $DToroid_C$ (blue)

$CSDC$ also cuts $DBlowup_A$ into regions, inside and outside $CSDC$ (black)

Applying Basic Algebraic Geometry



Top-Down View of $DBlowup_A$

Inside versus outside $CSDC$ is easily seen

The half that is outside $Toroid_A$ is labeled "A:0", while the half that is inside $Toroid_A$ is labeled "A:1"

$DToroid_B$ cuts $DBlowup_A$ into a portion outside $Toroid_B$ (labeled "B:0"), etc.

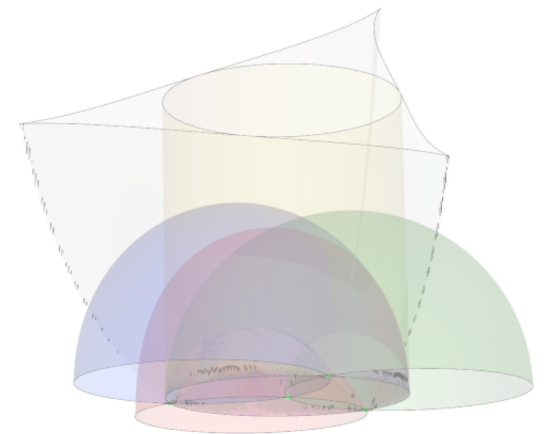
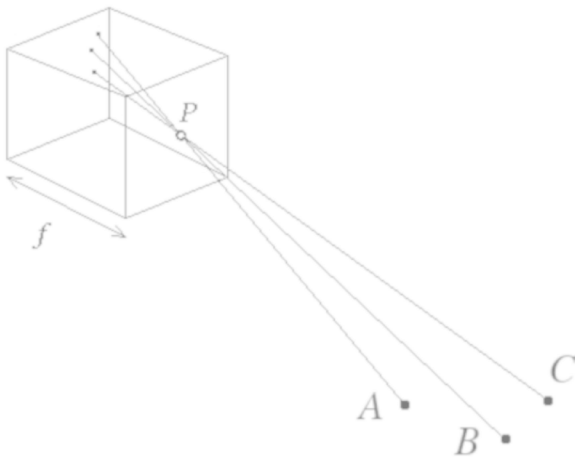
Similarly for $DToroid_C$

Counting and Finding Related Points

Given a point in 3D real space, the problem of determining how many strongly related points it has, and where they are located, is really quite difficult

Steady headway has been made in the past couple decades, and the following presents state-of-the-art results

As hinted at earlier, smoothly moving particles are employed for this purpose

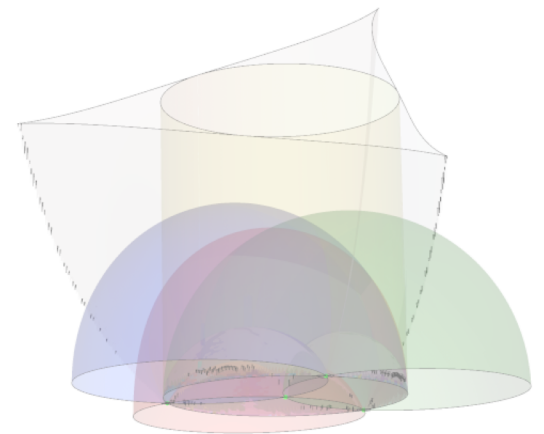
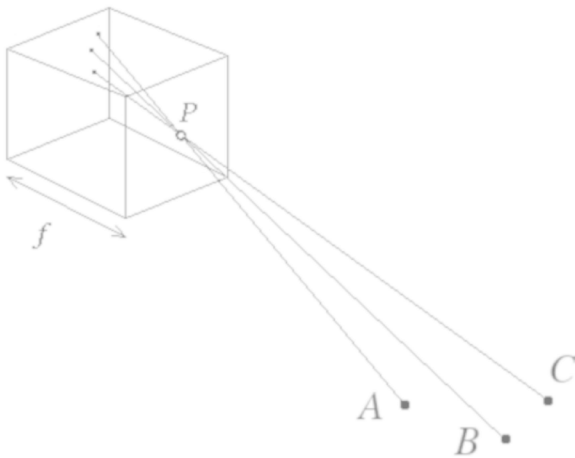


Counting and Finding Related Points

To avoid counting duplicates, *i.e.* reflections about the xy -plane, we will focus our attention solely on the upper half-space ($z > 0$)

Whenever a particle should pass smoothly through a control point, we can regard it instead as bouncing off (non-smoothly) the control point

Recall that strongly related points have the same view angle, and hence are always in the same toroidal region



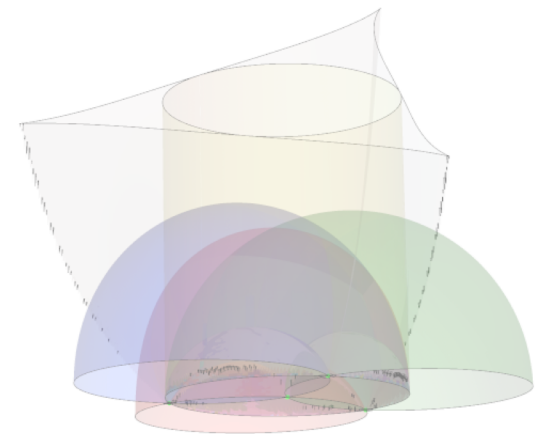
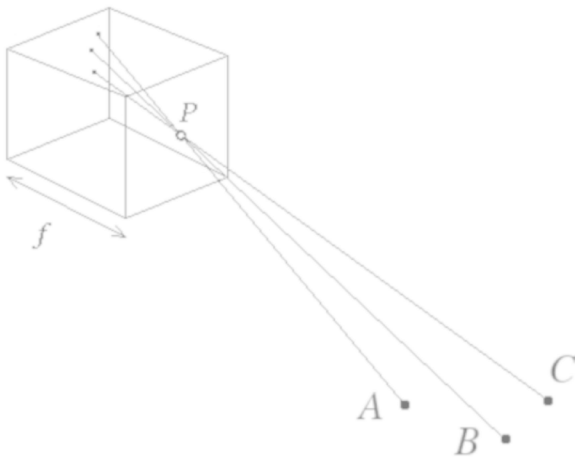
Counting and Finding Related Points

What about the weak relatives?

The analysis of this begins with simple observations such as this: if a particle p is in region 110, then any weak relative q must be either in regions 110 or region 112

If p and q are both in region 110, then it is possible that all three of their view angles are the same, in which case, p and q are strongly related

But it is also conceivable for them to have supplementary values for angles α and β

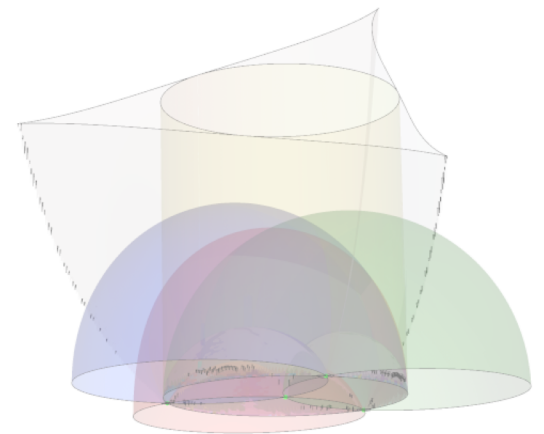
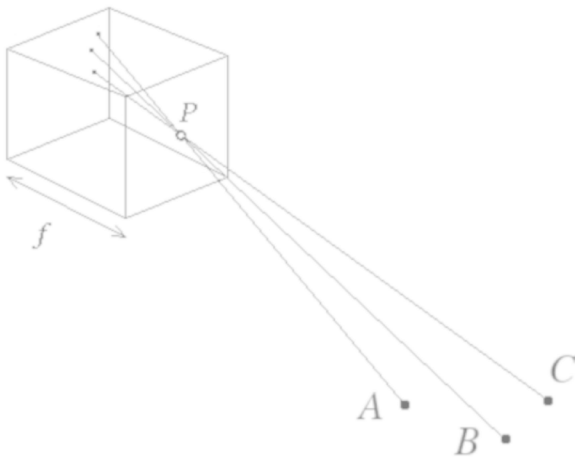


Counting and Finding Related Points

A simple result to state, but somewhat tricky to prove is this:

If a particle is outside *CSDC*, then, in the upper half-space, it has exactly one weak relative, but when it is inside the *CSDC*, it has exactly three weak relatives

But a point on the danger cylinder counts twice here

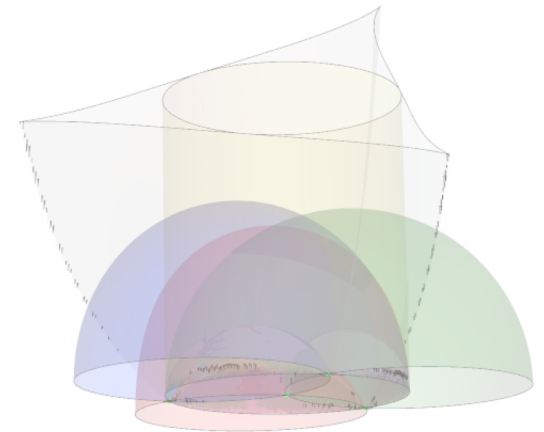
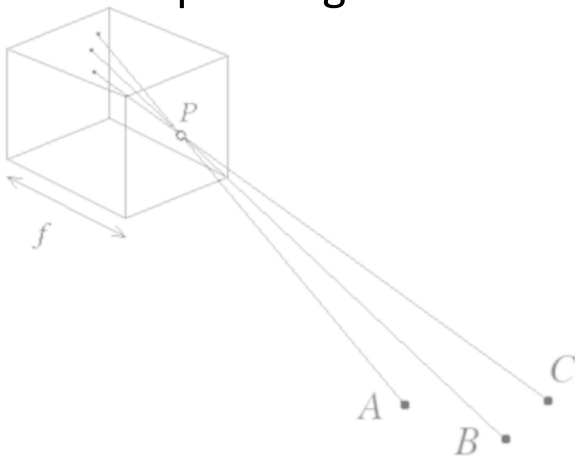


Counting and Finding Related Points

Using the the transition graph and previous results, the following facts emerge

Inside the *CSDC*, four weakly related points in the upper half-space must be in one of the following combinations of regions: $(000, 000, 000, 000)$, $(100, 100, 100, 122)$, $(110, 110, \underline{112}, \underline{112})$, $(111, \underline{111}, \underline{111}, \underline{111})$, or a quadruple obtained from these by permuting the entries in the triples together, and/or permuting the triples in a quadruple

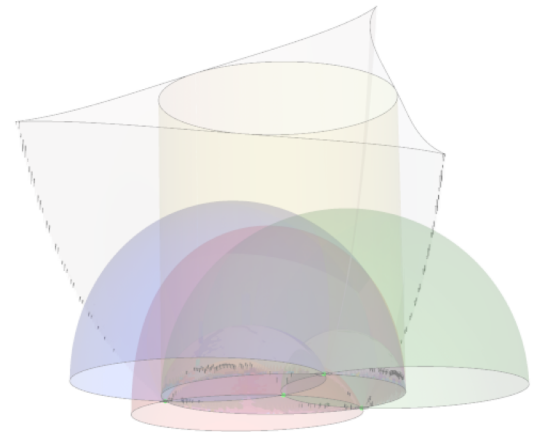
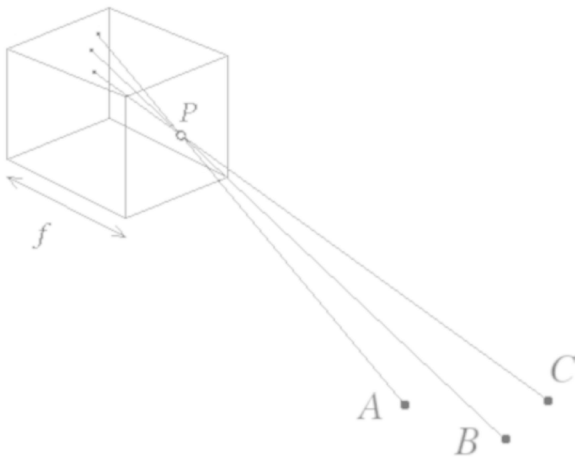
An underlined 1 here means that the angle involved is *supplementary* to the angle corresponding to a non-underlined 1



Counting and Finding Related Points

Outside the *CSDC*, two weakly related points in the upper half-space must be in one of the following combinations of regions: $(000, 000)$, $(100, 122)$, $(\underline{112}, 1\underline{12})$, $(110, 110)$, $(111, 1\underline{11})$, or a pair obtained from these by permuting the entries in the triples together, and/or swapping the triples in a pair

The above results lead almost immediately to a proof of the following result, which was conjectured by **Bo Wang**:

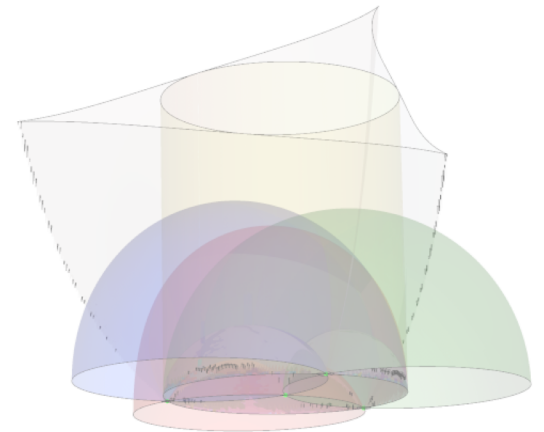
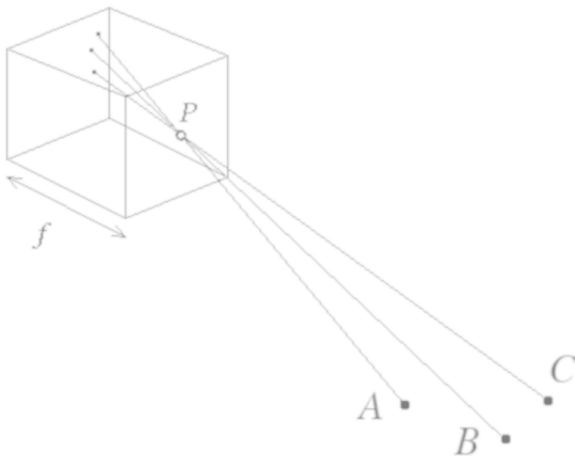


Counting and Finding Related Points

Theorem

Assume the control points triangle is acute. Then,

- (i) Inside the *CSDC*: a point inside precisely 0, 1, 2, 3 of the basic toroids has precisely 3, 2, 1, 0 strongly related (real) points, respectively
- (ii) Outside the *CSDC*: a point inside precisely 0, 1, 2, 3 of the basic toroids has precisely 1, 0, 1, 0 strongly related (real) points, respectively

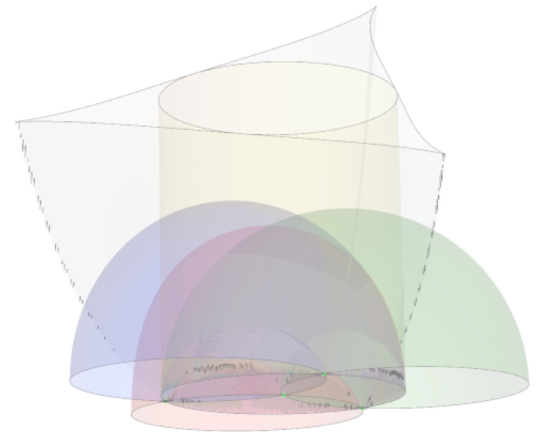
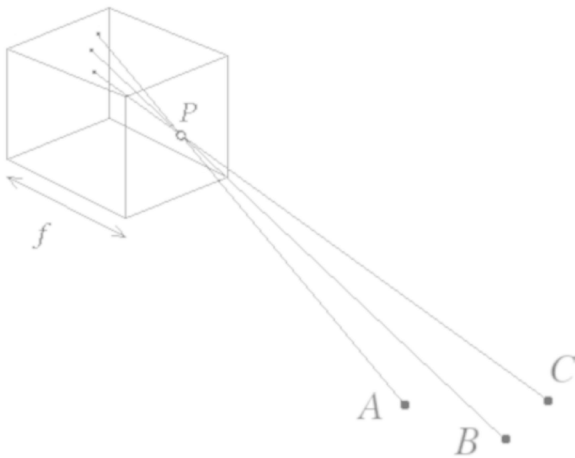


More Algebraic Geometry

All of three-dimensional real space can be smoothly enlarged, by blowing up a point, say A , as follows

A standard way to blow up the point A is to consider the set of all ordered pairs (p, l) , where p is a point in 3D real space, and l is a line through p and A

This set of ordered pairs can naturally be given the structure of a variety (space)



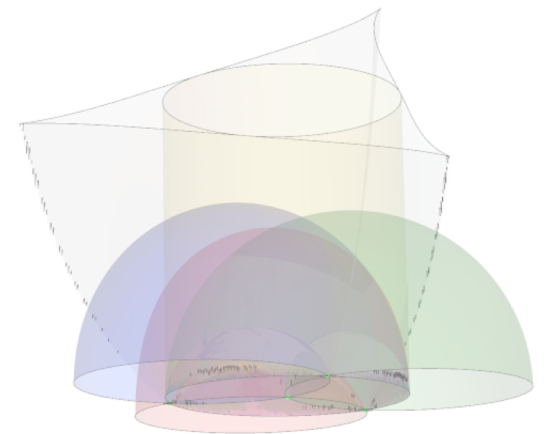
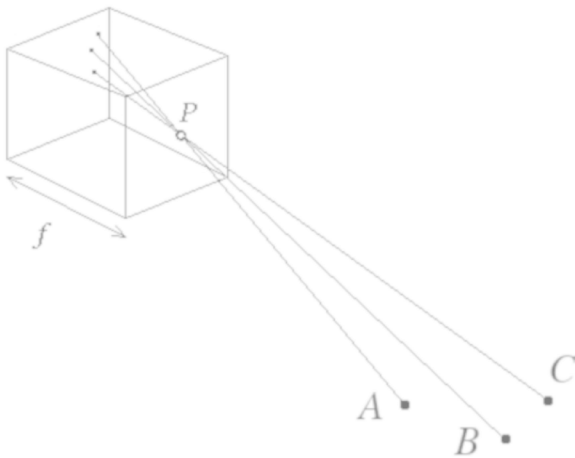
More Algebraic Geometry

Each point p other than A corresponds to a unique pair (p, l)

But there are infinitely many pairs of the form (A, l) because every line l through A is allowed here

In the new variety, A has been replaced with infinitely many new “points”

We have blown up the point A !

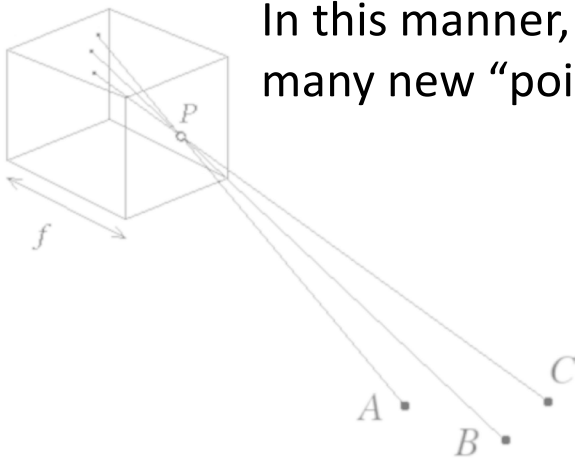


More Algebraic Geometry

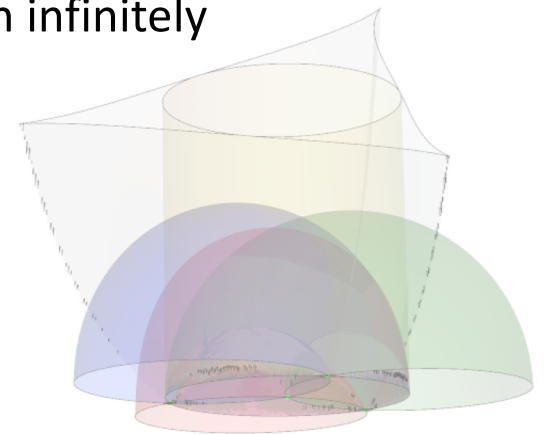
For the purpose of better studying P3P, it is useful to blow up all three control points, A, B and C

This amounts to a straightforward embellishment of what we just did

Instead of all possible (p, l) , use the set of all possible (p, l_A, l_B, l_C) , where l_A is a line through p and A, l_B is a line through p and B, and l_C is a line through p and C



In this manner, each of A, B and C gets replaced with infinitely many new “points” (a whole projective plane)

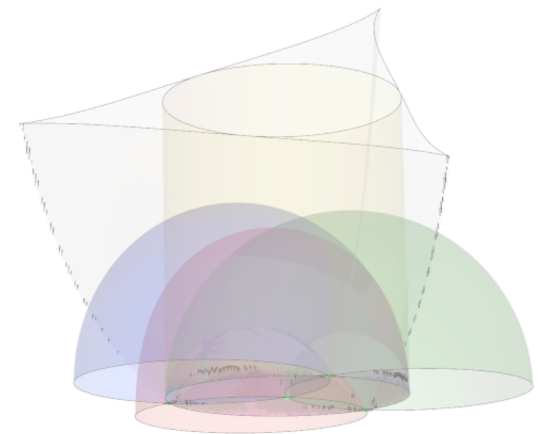
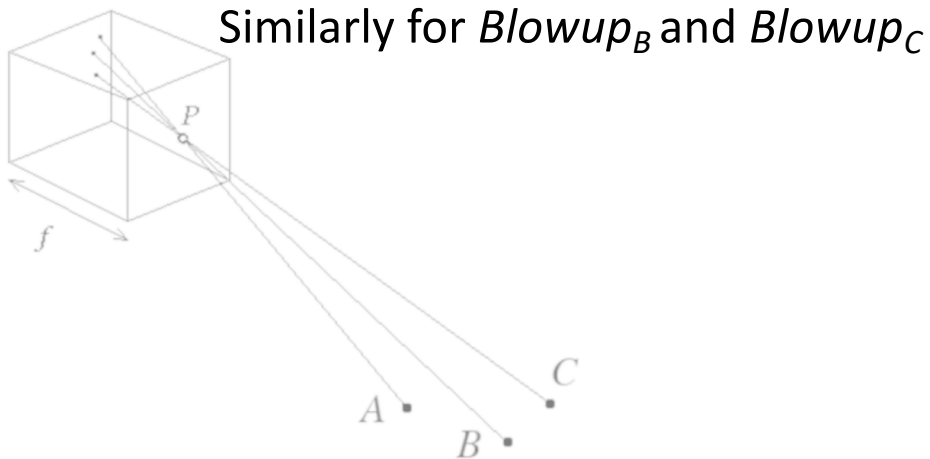


More Algebraic Geometry

Let *Blowup* denote our new space (variety) of all suitable quadruples (p, l_A, l_B, l_C) , and refer to the quadruples as its “points”

Each point of the form (A, l_A, l_B, l_C) is only distinguished by l_A , which we visualize as specifying a tangent line when we travel through the point A in xyz-space, etc.

These points constitute a subvariety of *Blowup* that we will identify with *Blowup*_A

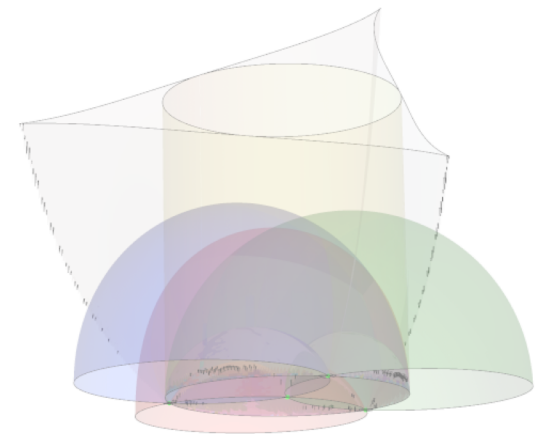
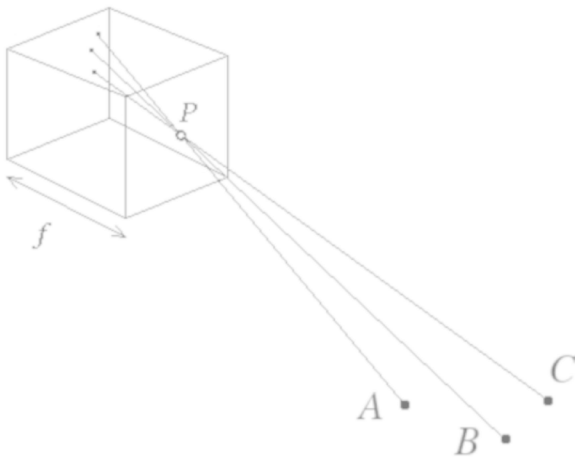


More Algebraic Geometry

Next, for any variety V in 3D real space, such as one of our double toroids, we can obtain a corresponding variety in *Blowup*, as follow:

1. Each point of V , other than A , B or C , corresponds to a unique point in *Blowup*
2. Take all of these points in *Blowup*, and extend this set by including any of its limit points (topological closure)

This will be a subvariety of *Blowup*, and any limit points that might get “tossed in” will come from the union of $Blowup_A$, $Blowup_B$ and $Blowup_C$

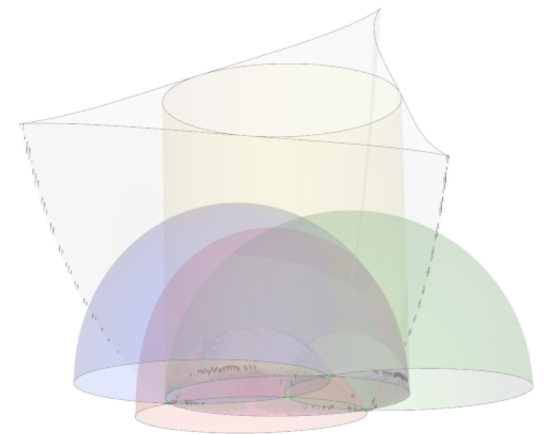
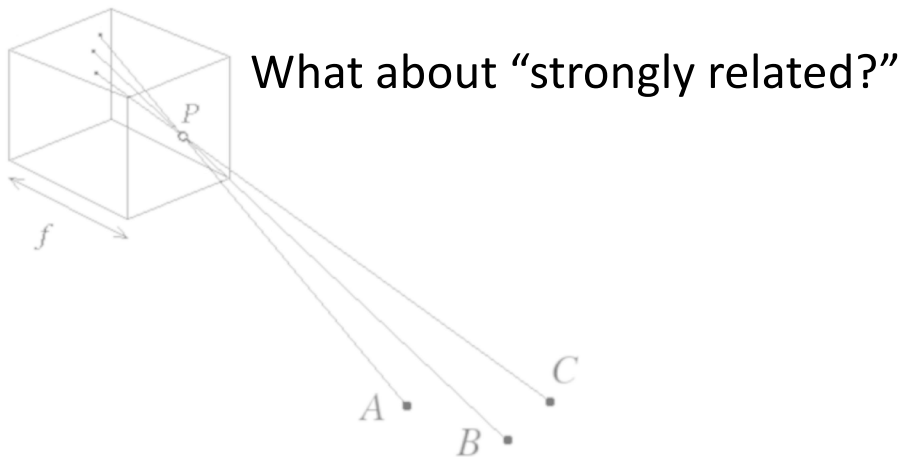


More Algebraic Geometry

Here is one reason for using the variety *Blowup* in connection with P3P: the important quantities c_1^2 , c_2^2 , c_3^2 and $c_1c_2c_3$ all extend naturally to all of the points of *Blowup*

There is a nice map from *Blowup* onto the variety V of all points (q, r, s, t) for which $q^2 = rst$ that sets $q = c_1c_2c_3$, $r = c_1^2$, $s = c_2^2$ and $t = c_3^2$

Points of *Blowup* are weakly related if and only if they have the same image under this map from *Blowup* to V , so “weakly related” becomes an algebraic geometric concept



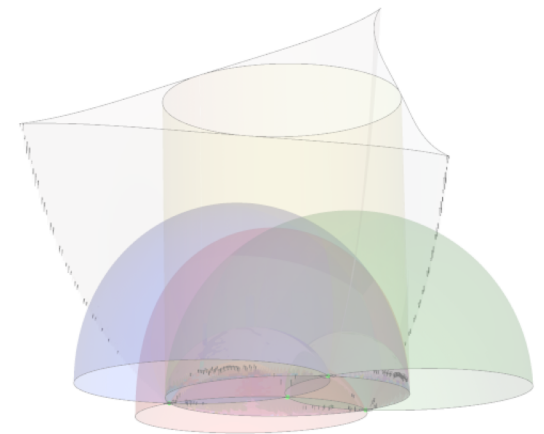
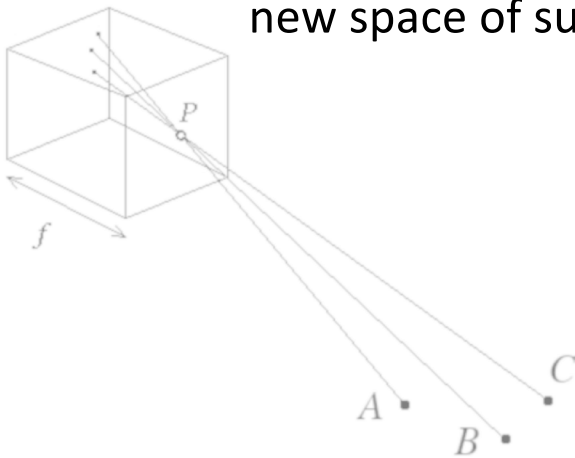
More Algebraic Geometry

Unfortunately, c_2 and c_3 are not defined on $Blowup_A$, and so forth, and this is basically why on $Blowup$, we have continuity of “weakly related,” but not “strongly related”

One way to “correct” for this is a bit messy, and amounts to replacing the lines l_A, l_B and l_C in (P, l_A, l_B, l_C) with *rays* from P that either point towards or away from the corresponding control point

Of course, α, β and γ are then defined to be the angles between pairs of these rays, etc.

This means each ordinary point P in 3D real space gets replaced by **eight** points in the new space of such quadruples $(P, \iota_A, \iota_B, \iota_C)$ -- *Ouch!*

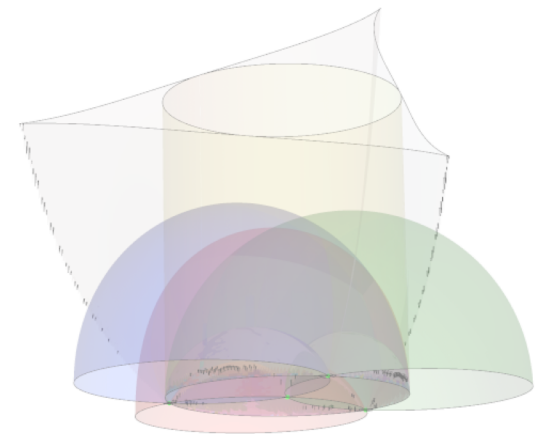
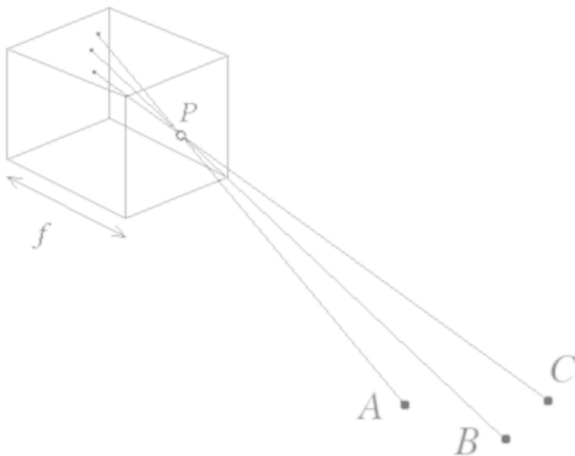


More Algebraic Geometry

We see that the variety Blowup^* thus constructed is an 8-fold cover of Blowup

We can imagine obtaining this space from eight copies of xyz-space, by extracting tiny open balls around A, B and C, from each copy, and gluing these together appropriately along the spherical boundaries of the balls

The 8-fold cover can easily be reduced to a 4-fold cover by identifying each point $(P, \mathbf{r}_A, \mathbf{r}_B, \mathbf{r}_C)$ with the point $(P, -\mathbf{r}_A, -\mathbf{r}_B, -\mathbf{r}_C)$, where $\mathbf{r}_A$ and $-\mathbf{r}_A$ are oppositely directed rays, defining the same undirected line, and so forth



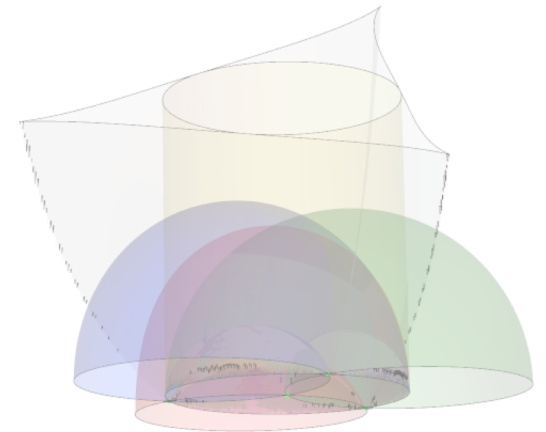
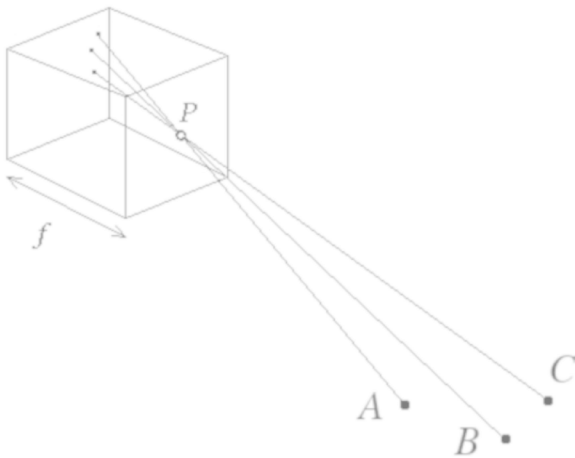
More Algebraic Geometry

Denoting this 4-fold cover by $Blowup^\#$, we have a variety on which the quantities c_1 , c_2 and c_3 can be naturally defined (the same may be said of $Blowup^*$)

These quantities vary continuously as a particle moves continuously around in $Blowup^\#$

There is a natural projection from $Blowup^\#$ onto $Blowup$

A continuous open path in $Blowup^\#$ may project onto a closed path in $Blowup$, thus showing that $Blowup^\#$ is a non-trivial cover of $Blowup$, in the sense of *homotopy*

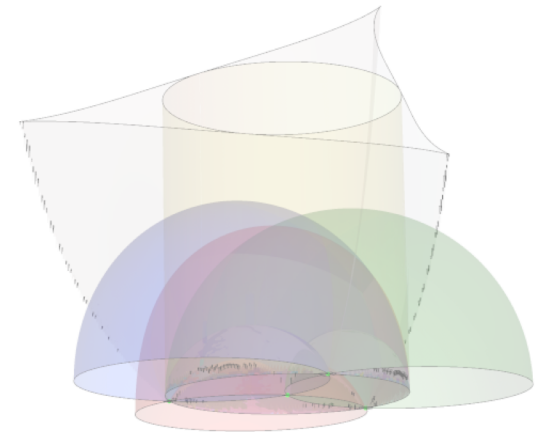
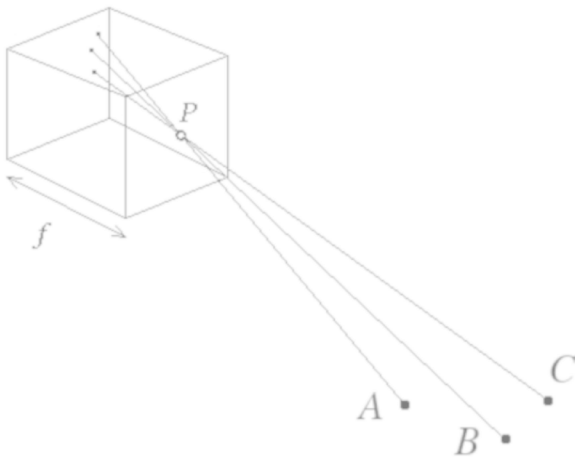


More Algebraic Geometry

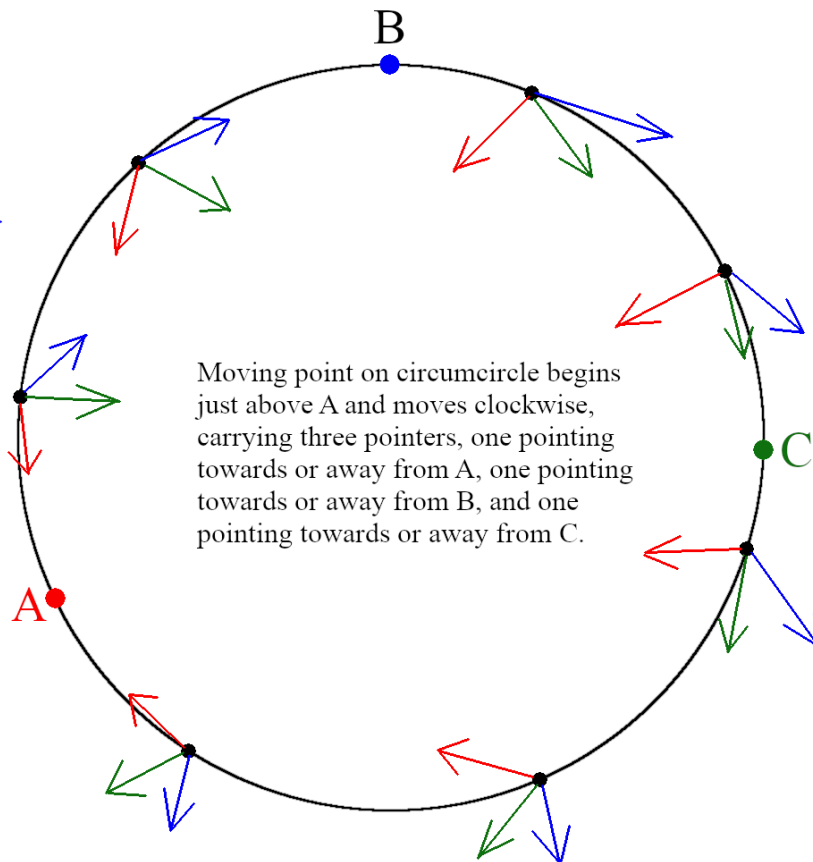
It can be shown that $\text{Blowup}^\#$ contains four copies of the circumcircle on which α , β and γ are constant

By analogy with a 2D version of P3P (Maienschein and Rieck), it makes sense to collapse each of these circles to a single point (via a “blowdown,” the opposite of a blowup)

This yields four special points, and we will denote the resulting space $\text{Blowup}^\dagger$, and observe that (α, β, γ) and (c_1, c_2, c_3) are naturally defined here as well



More Algebraic Geometry



Representing a lift of the circumcircle to $Blowup^*$ or $Blowup^\#$

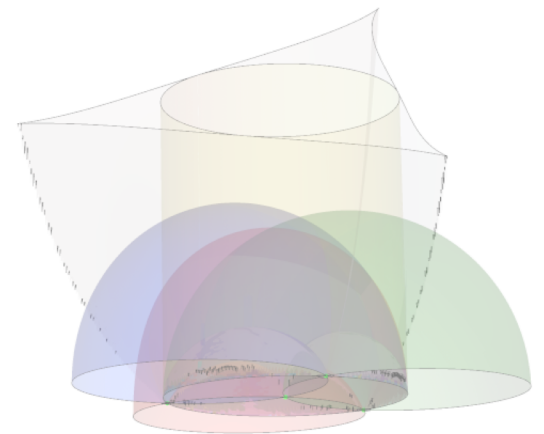
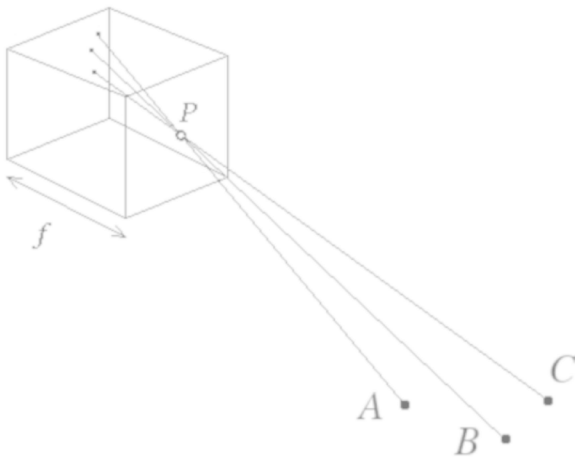
When working in $Blowup^*$, a lift of a moving particle on the circumcircle must go twice around the circle to return to its starting position

When working in $Blowup^\#$ instead, only one trip around the circle is needed

The circumcircle thus lifts to a circle in $Blowup^\#$, in fact *four* such circles, which can be collapsed to make $Blowup^+$

Closing Questions

1. Why does a deltoid curve occur in the limiting case as $|z|$ goes to infinity?
2. Why does the deltoid formula occur in a nice compact way of writing the otherwise extremely messy discriminant formula for Grunert's system of equations?
3. Is there a simple way to visualize the variety that results from blowing up the three control points, and related varieties?



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Slides available at mqriek.com/P3P/BlowingUpControlPoints.pptx , [..._audio.pptx](#) and [...pdf](#)

Thank You

Questions?