Blowing up Control Points

Algebric Geometry and Topology Applied
to the Perspective Three-Point Problem
by

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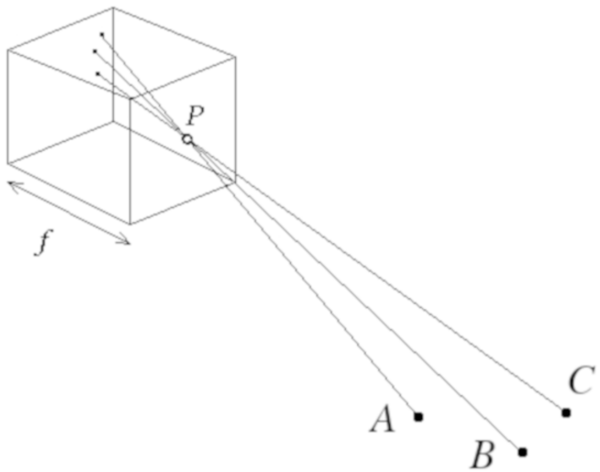

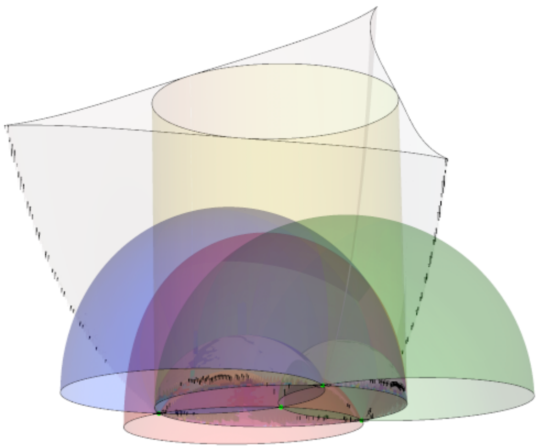

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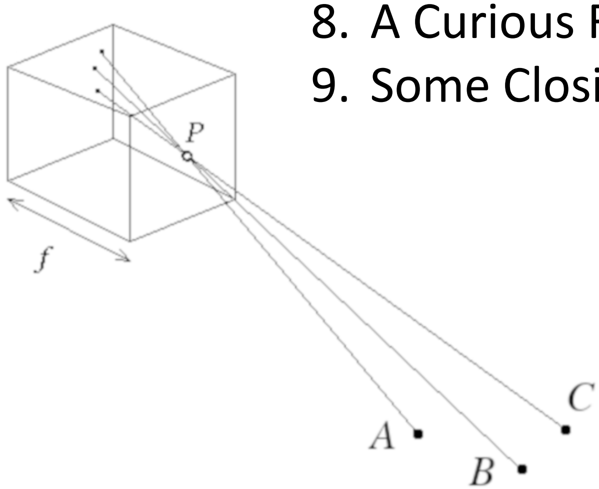
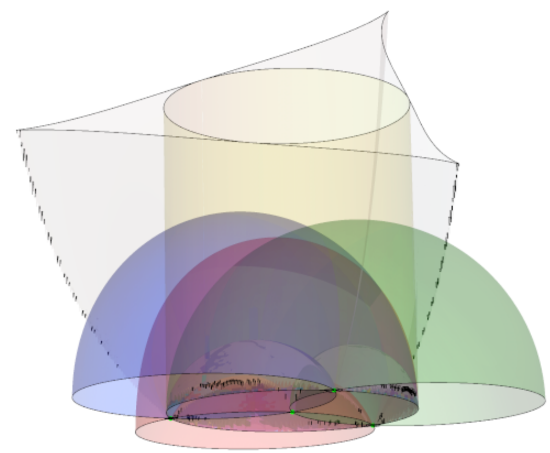
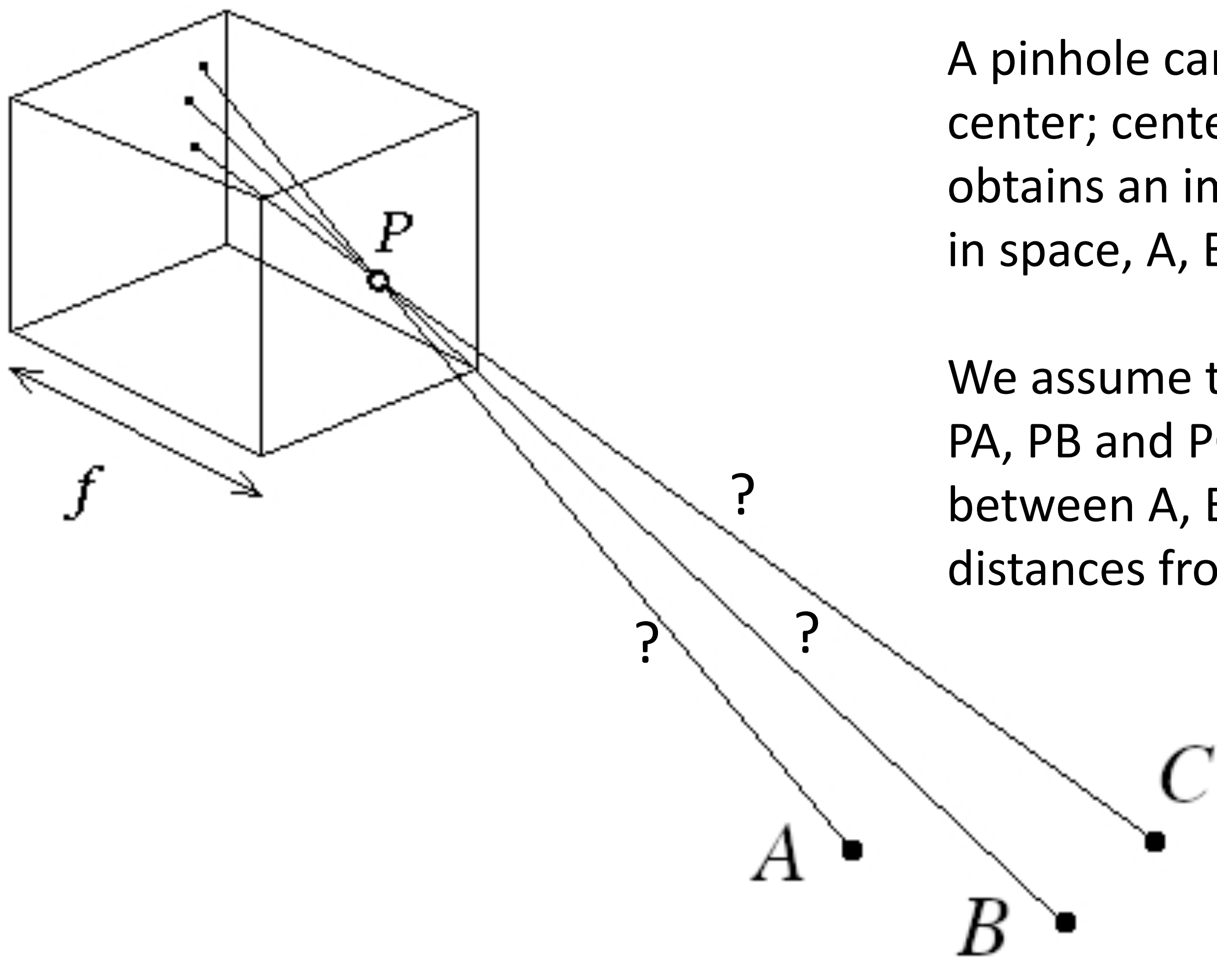




An Introduction to the P3P Problem



A pinhole camera, with *pinhole* (optical center; center of perspective) located at P, obtains an image showing three known points in space, A, B and C, called the *control points*

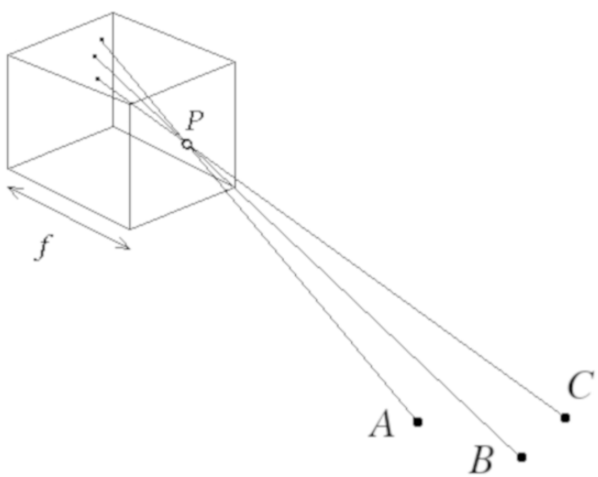
We assume that the angles between the rays PA, PB and PC are known, the distances between A, B and C are known, but the distances from P to A, B and C are *unknown*

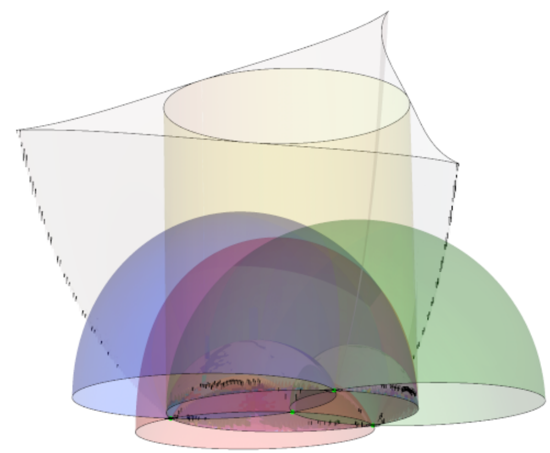
An Introduction to the P3P Problem

The primary goal of the P3P problem is to determine the distances from the camera to the control points, and thereby determine the cameras position (in relation to A, B and C)

Let α, β and γ be the known angles between rays PB and PC, between rays PC and PA, and between rays PA and PB, respectively

We will call α, β and γ, the *viewing angles*

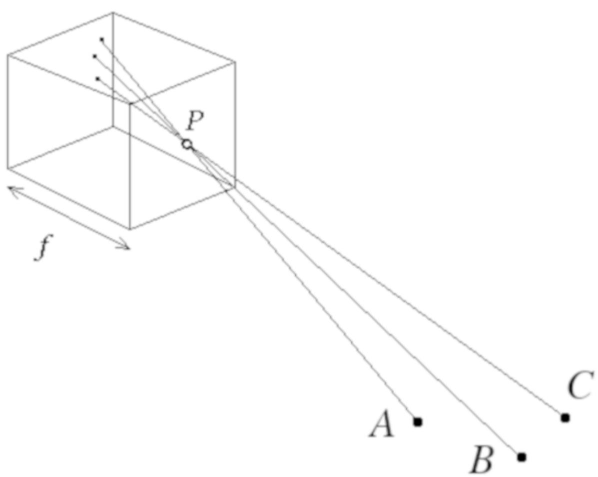


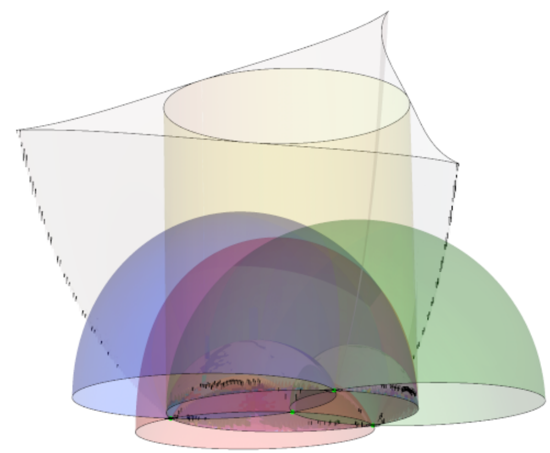


Let $c_1 = \cos\alpha$, $c_2 = \cos\beta$ and $c_3 = \cos\gamma$

Let r_1, r_2 and r_3 be the *unknown* distance from P to the control points A, B and C, respectively

Let d_1 be the known distance between B and C, let d_2 be the known distance between C and A, and let d_3 be the known distance between A and B

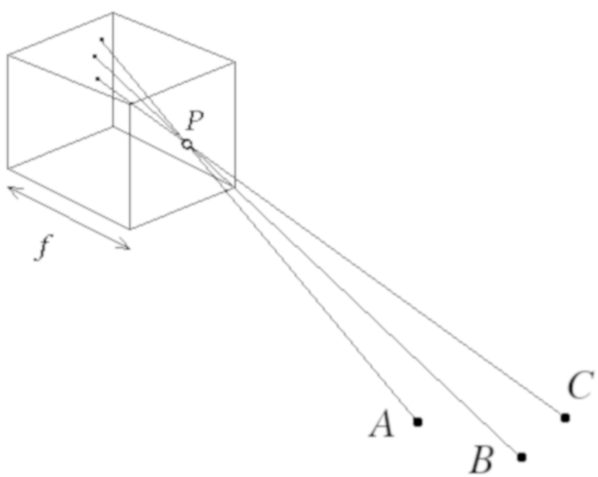


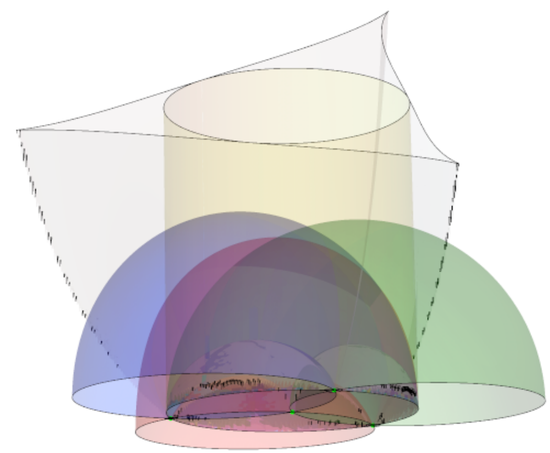


An Introduction to the P3P Problem

Using the Law of Cosines, we obtain *Grunert's system* of equations, where r_1 , r_2 and r_3 are the only unknowns:

$$\begin{cases} r_2^2 + r_3^2 - 2\, c_1\, r_2\, r_3 = d_1^2 \\ r_3^2 + r_1^2 - 2\, c_2\, r_3\, r_1 = d_2^2 \\ r_1^2 + r_2^2 - 2\, c_3\, r_1\, r_2 = d_3^2 \end{cases}$$



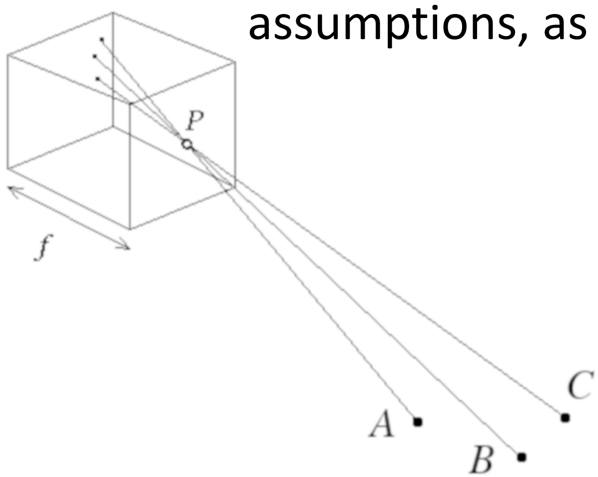


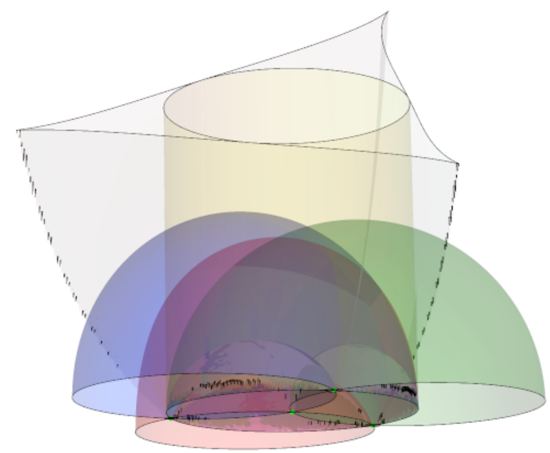
An Introduction to the P3P Problem

Over the years, this system has been solved in various ways, usually by eliminating two of the unknowns, so as to produce a fourth-degree (quartic) polynomial equation in some unknown quantity

Yet significant questions persist, even after many decades of study of P3P

In recent years, the analysis of P3P has benefited substantially from algebraic manipulation software, and from making a few harmless mathematical assumptions, as follows



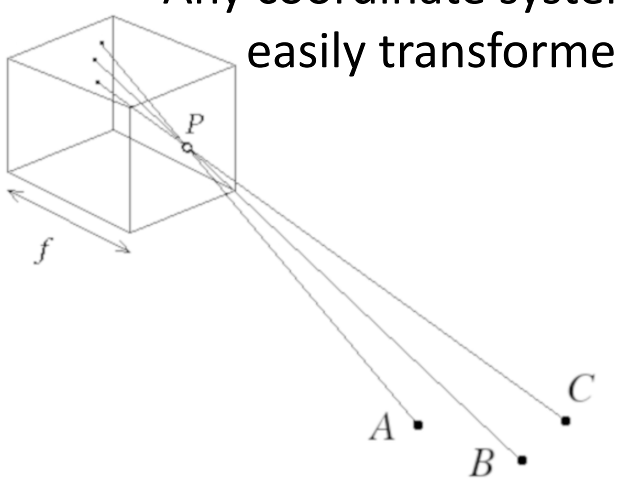


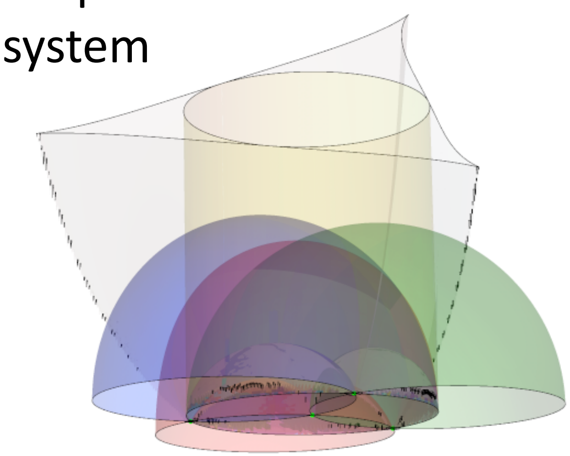
An Introduction to the P3P Problem

An xyz-coordinate system (respecting the actual angles) is used so that the control points all lie on the unit circle in the xy-plane

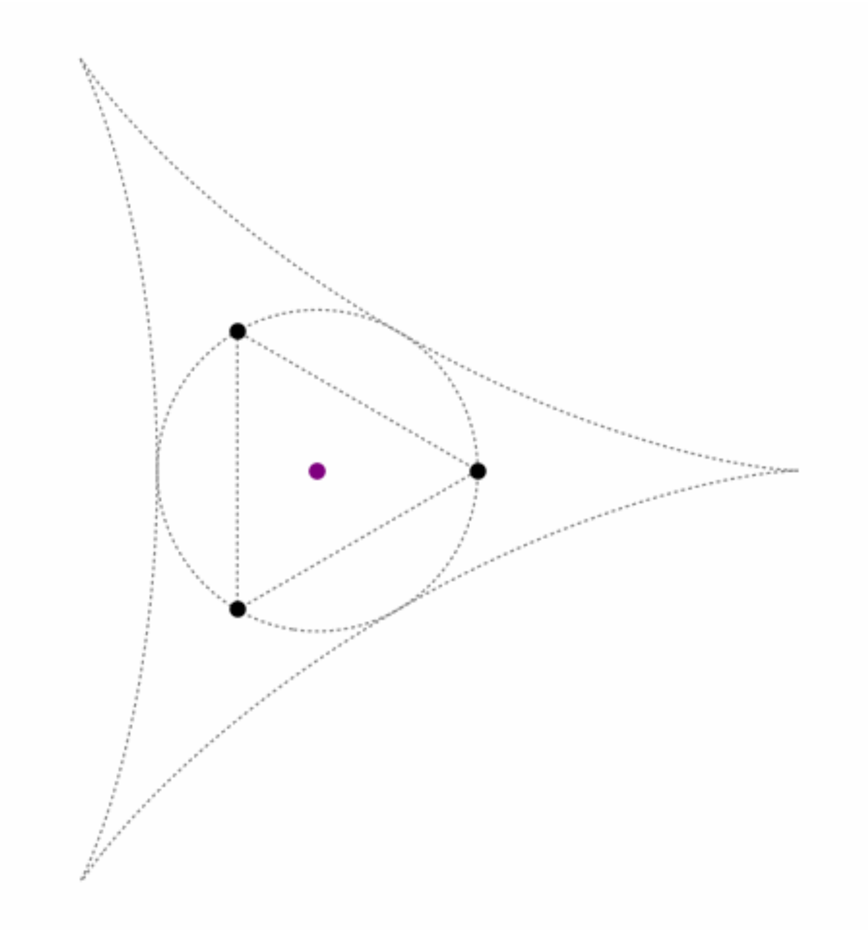
Letting $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ and $(x_3, y_3, 0)$ be the coordinates of A, B and C, respectively, assume further that there are angles ϕ_1 , ϕ_2 and ϕ_3 such that $x_i = \cos \phi_i$ and $y_i = \sin \phi_i$, for i = 1, 2, 3, and such that $\phi_1 + \phi_2 + \phi_3 = 0$

Any coordinate system that respects the angles in physical space can be easily transformed to produce a suitable coordinate system





An Introduction to the P3P Problem



(Wait for animation)

It is henceforth assumed that the control points are on the unit circle in the xy-plane, with a particular orientation, as prescribed in the previous slide

Consider the control points to be the vertices of a triangle

The triangle's orthocenter (purple dot) stays inside a particular *deltoid curve*

When the triangle is acute, the orthocenter will be inside the triangle

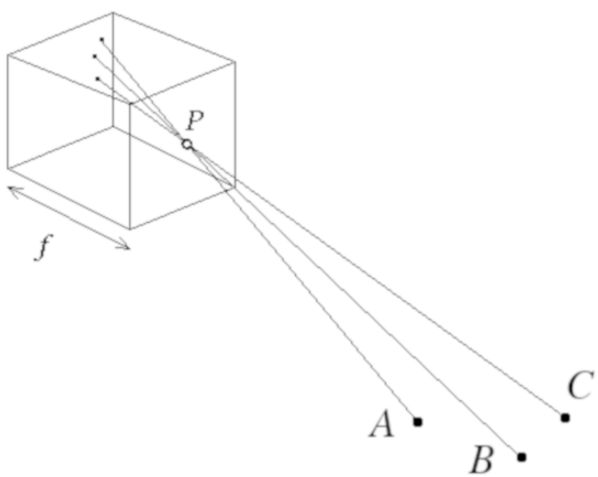
When the triangle is obtuse, the orthocenter will be outside the triangle's circumcircle (*i.e.* the unit circle)

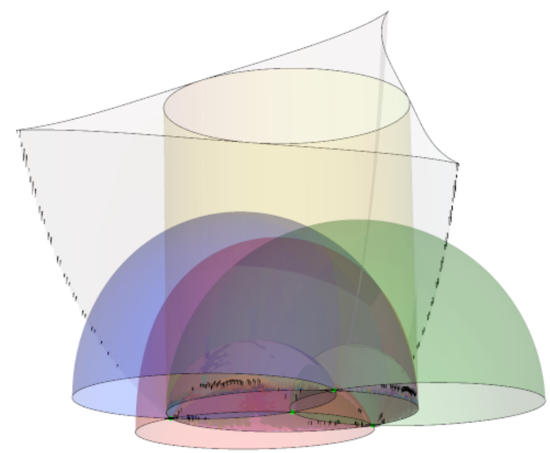
Despite the animation, the control points are assumed to be fixed points in space

An Introduction to the P3P Problem

Each point P in space, having Cartesian coordinates (x, y, z), except for the control points, is such that if the camera's pinhole happens to be at P, then the angles α, β, γ, and the cosines c_1 , c_2 , c_3 can easily be expressed in terms of x, y and z

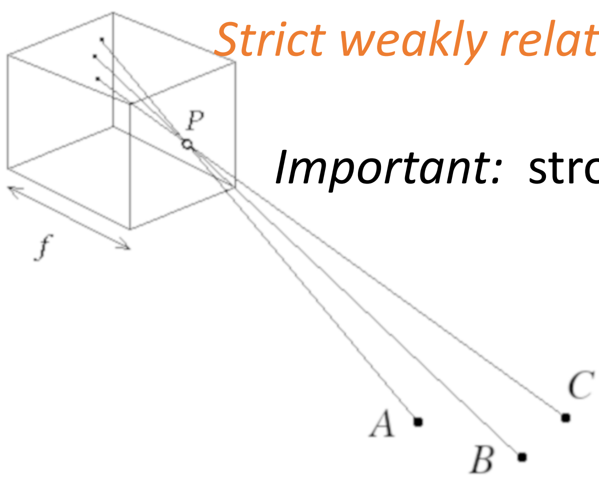
In this way, it is often useful to regard α, β, γ, c_1 , c_2 and c_3 as being given by functions of the Cartesian coordinates, x, y and z, for analytic purposes, even though x, y and z are unknown in the P3P Problem





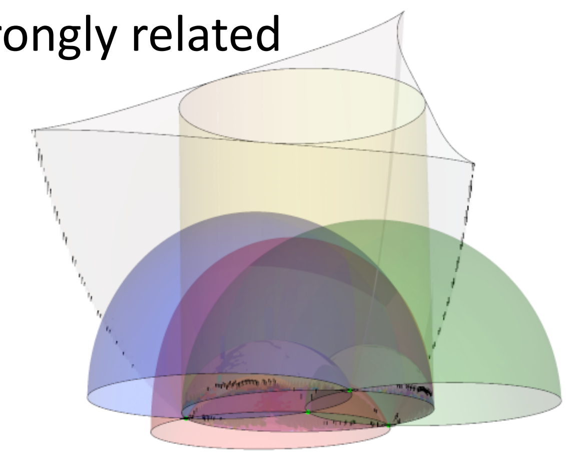
Two non-control points in xyz-space will be called *strongly related* if their Cartesian coordinates yield the same values for the c_1 , c_2 and c_3

Two non-control points in xyz-space will be called *weakly related* if their Cartesian coordinates yield the same values for the quantities c_1^2 , c_2^2, c_3^2 and $c_1 c_2 c_3$; in other words, they have the same values for $|c_1|$, $|c_2|$ and $|c_3|$, and disagree in an even number (0 or 2) of the signs of the cosines



Strict weakly related means weakly related but not strongly related

Important: strongly related implies weakly related

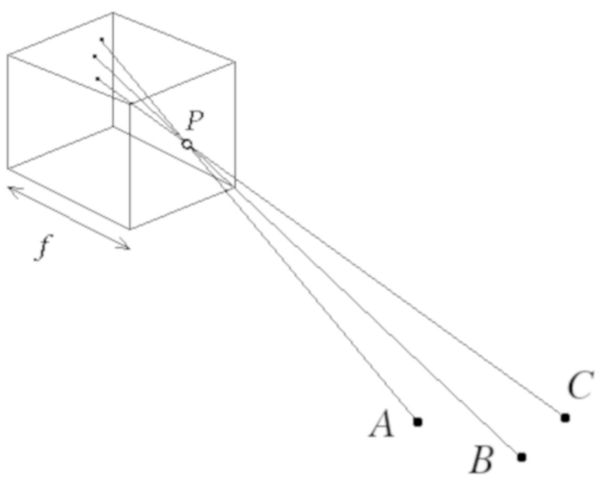


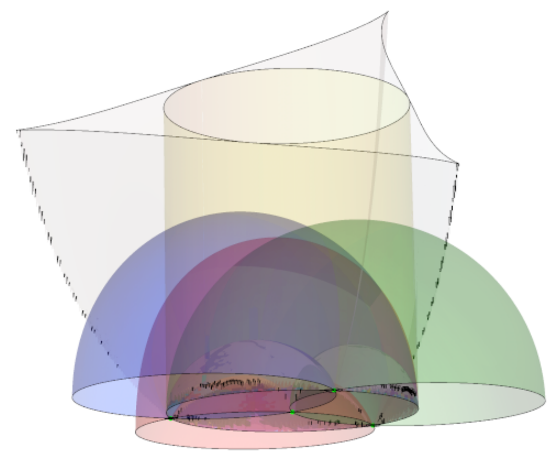
An Introduction to the P3P Problem

For given values of its *parameters* d_1, d_2, d_3, c_1, c_2 and c_3, Grunert's system has multiple solutions, yielding up to eight *solution points* in real xyz-space

These solution points are strongly related, and the reflection of each solution point about the xy-plane is also a solution point

Often, it is desirable to focus only on the solutions in the upper half-space ($z > 0$), and ignore their reflections in the lower half-space ($z < 0$)

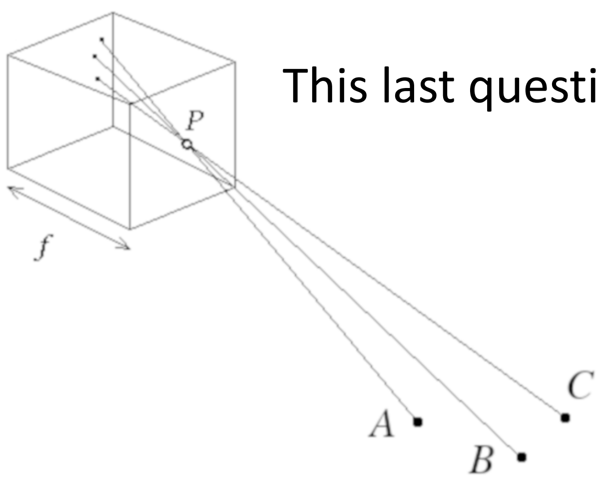




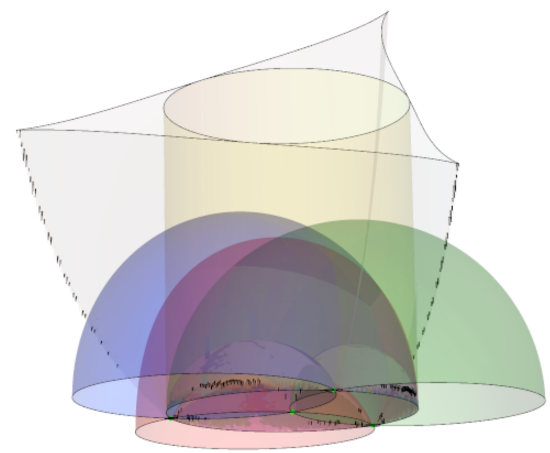
An Introduction to the P3P Problem

Here are a few interesting, and to different degrees, still outstanding, questions concerning P3P:

1. Given the P3P parameters d_1, d_2, d_3, c_1, c_2 and c_3, how many (real) solution points does Grunert's system yield?
2. Given a non-control point in xyz-space, how many strongly related points does it have (*i.e.* other solution points for the same P3P parameters)?
3. Where are they located?



This last question has been particularly elusive!

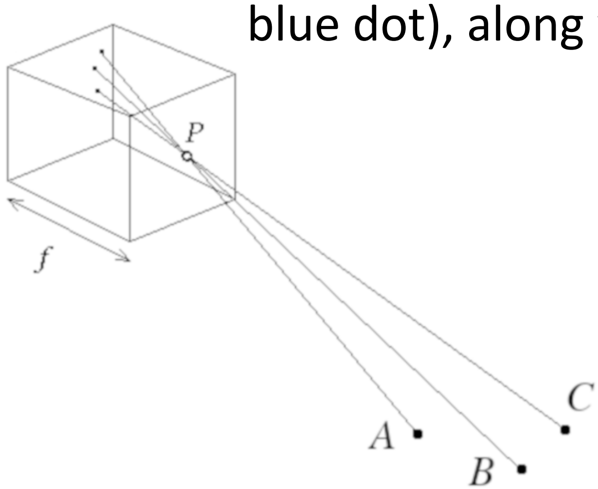


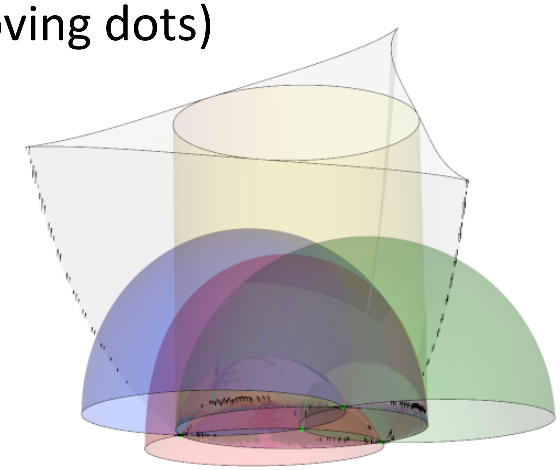
An Introduction to the P3P Problem

As a practical matter, strongly related points are of importance to real world problems, like camera tracking

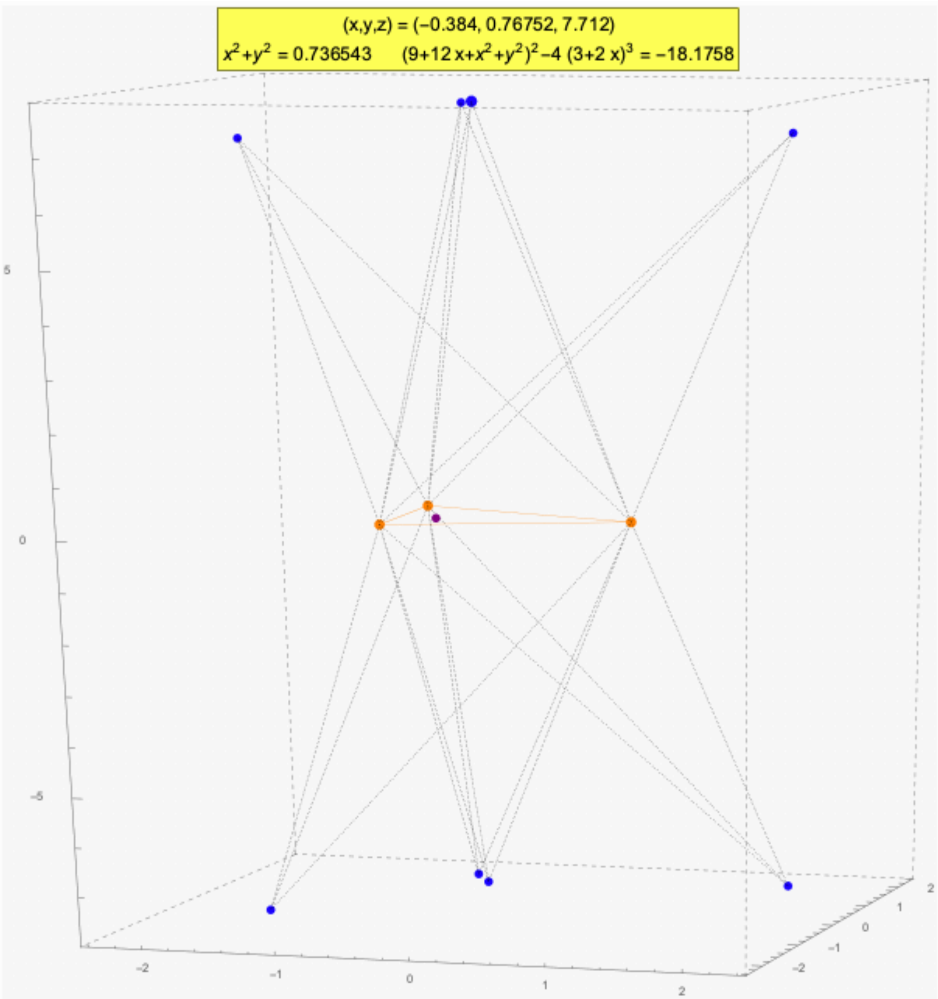
Until recently, there has been no interest in, or even discussion of, weakly related points, but as we will now see, the notion of “weakly related” exhibits better mathematical behavior, from the standpoint of continuity

The following animation shows the movement of a camera in space (the big blue dot), along with its related points (the other moving dots)





An Introduction to the P3P Problem



Related P3P Solution Points

(Wait for animation)

The orange dots are control points

The purple dot is the orthocenter

All of the blue dots are strongly related to each other

One of these, the biggest blue dot, is presumed to be the camera pinhole's position, whose coodinates can be seen at the top

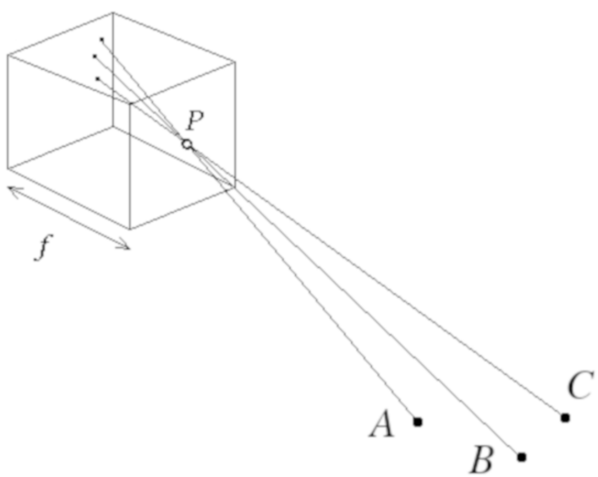
The green dots are only weakly related to the blue dots

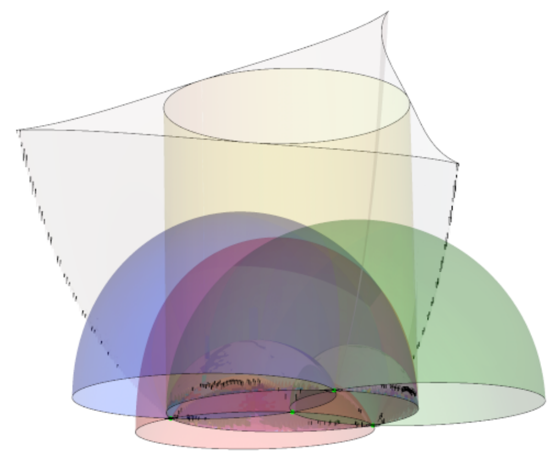
An Introduction to the P3P Problem

As the camera's pinhole moves continuously around in xyz-space, its weakly related points (including its strongly related points) move around mostly without being impeded, though they might pass through control points

In passing through a control point, such a point can switch between being strongly related and being strict weakly related (to the pinhole)

However, at certain locations (to be discusses later) weakly related points can come together and then disappear together (due to complex solutions)



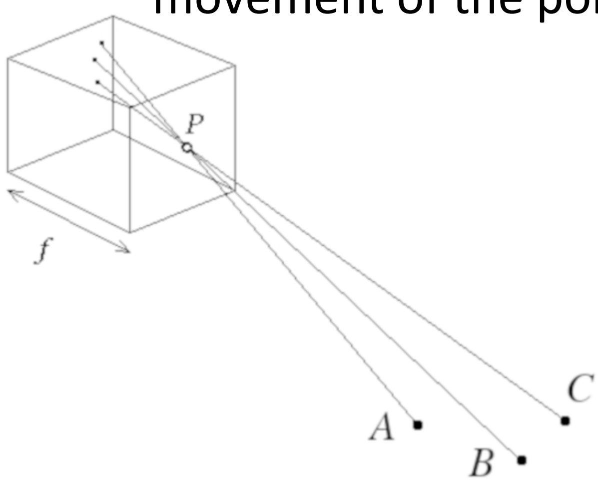


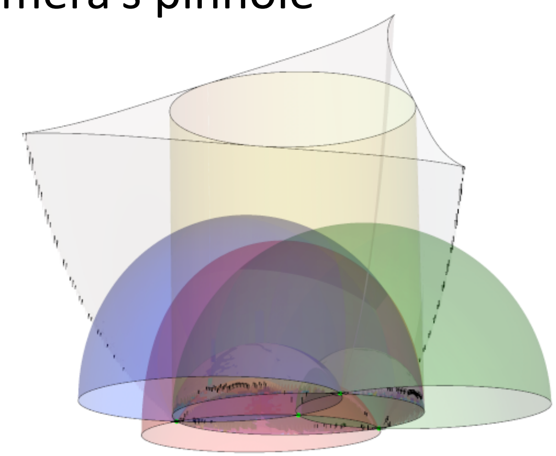
An Introduction to the P3P Problem

At the moment that one of the camera pinhole's weakly related points moves to a control point, it is technically not correct to say that the two moving points are still related, since only one of c_1, c_2 and c_3 is even defined at a control point

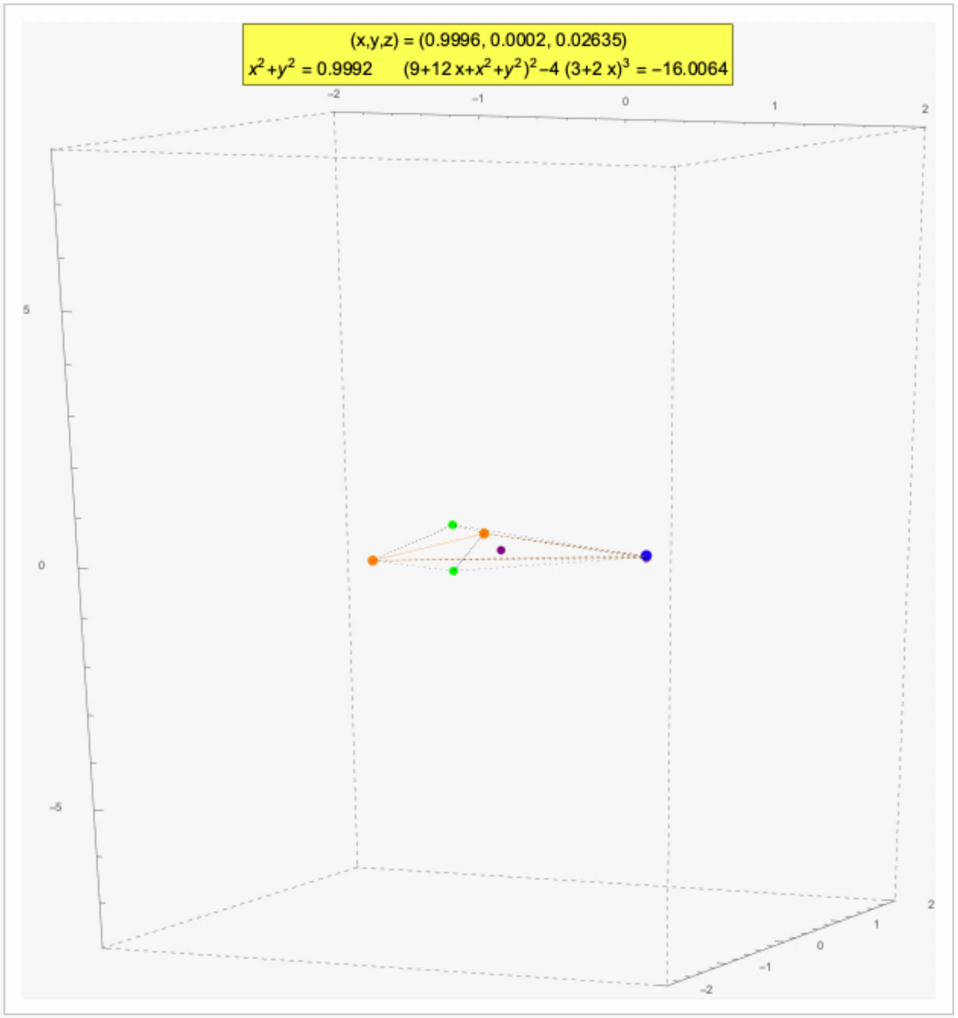
We will later use some algebraic geometry that corrects for this deficiency

Also, as seen in the next animation, if the camera's pinhole moves around, even slightly, in the vicinity of a control point, then this can cause substantial movement of the points that are weakly related to the camera's pinhole





An Introduction to the P3P Problem



Camera Pinhole Very Close to a Control Point

(Wait for animation)

When the camera's pinhole is close to one of the control points, even small movements of this pinhole will dramatically influence the weakly related points

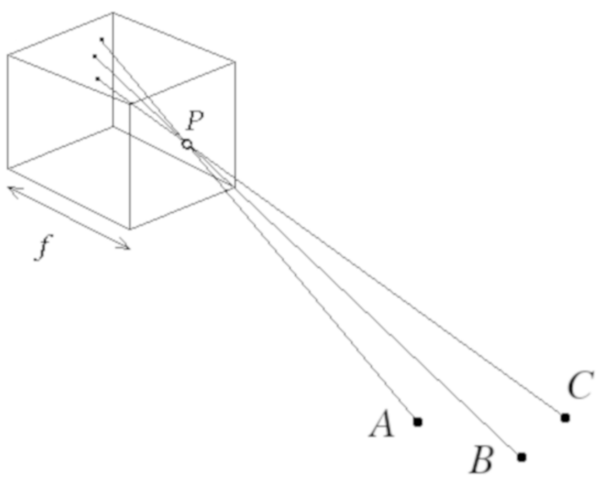
Notice the camera pinhole's coordinates, at the top, which are very close to the coordinates of a control point, namely (1, 0, 0), most of the time

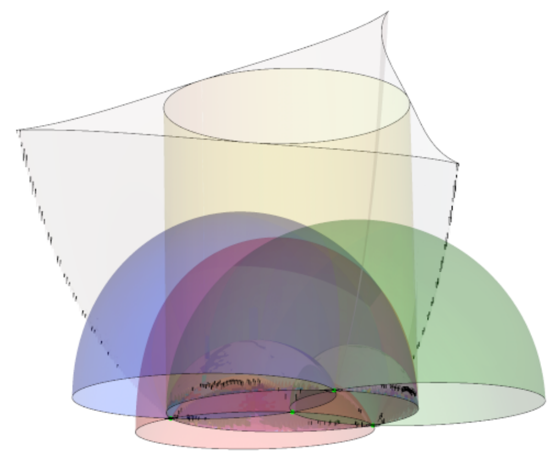
Toroids

The word "toroid" has multiple definitions, but here is the one that is useful for the P3P problem

A *toroid* is the surface that results in 3D (real) space, when an arc of a circle is rotated about the line through its endpoints

This surface is smooth, except at the two endpoints, where the surface is *singular*



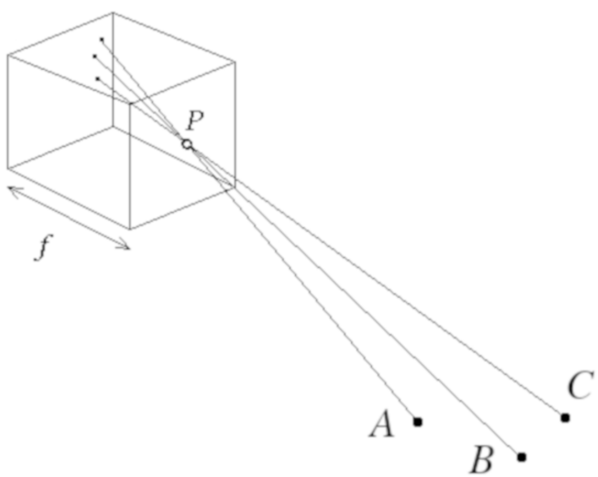


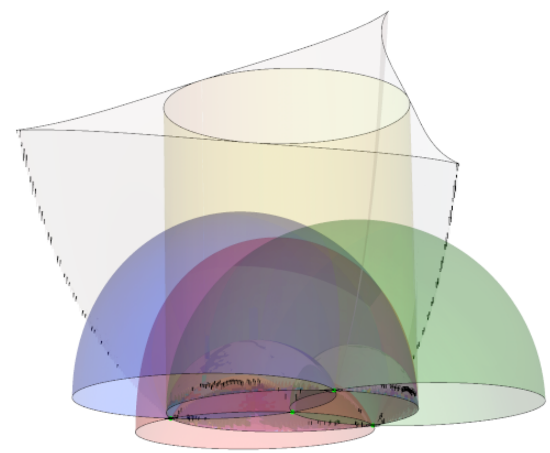
Toroids

Actually, the only toroids that will be useful for P3P always have two of the three control points as their endpoints

For example, take some point P in space, other that A or B, and consider the circular arc that passes through P, and that has A and B as its endpoints

Rotate this arc about the line AB to produce a toroid containing P





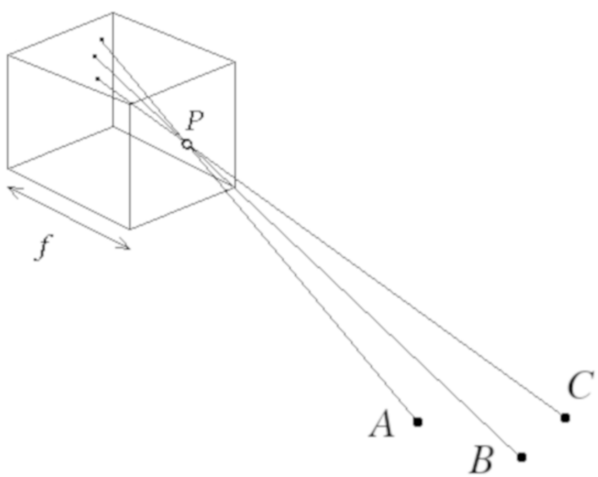
Toroids

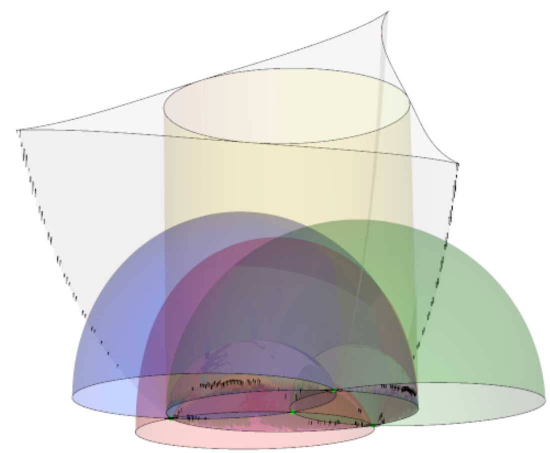
Question: Which points in space have the same γ value as P? (*I.e*. the angle between the two rays from the point, to A and to B, must be the same as at P)

Answer: The points on the toroid just described, except for A and B

The Inscribed Angle Theorem ensures that all of the points along the circular arc between A and B, and passing through P, have the same γ value as P

Rotating this arc about the line AB clearly does not affect the value of γ





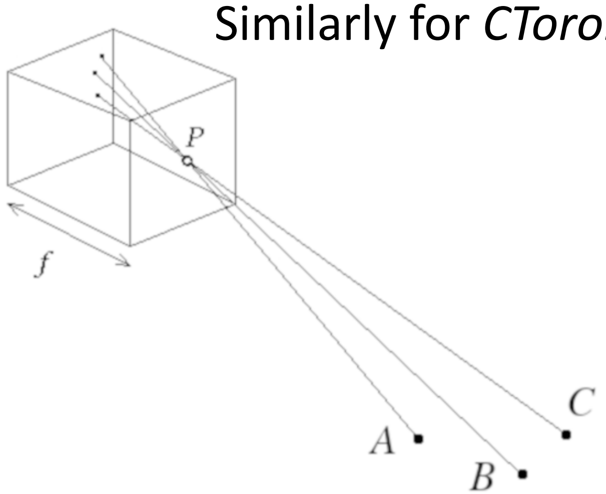
Toroids

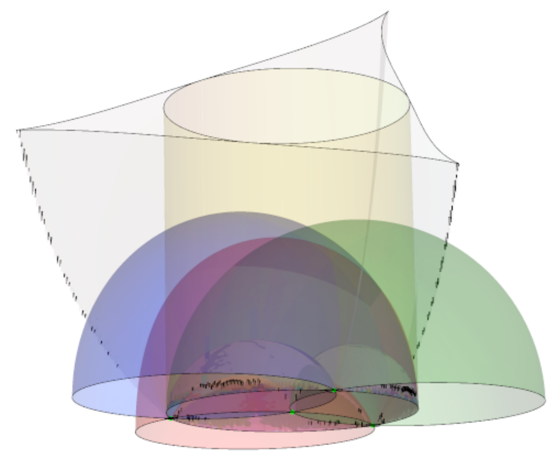
The toroid we just constructed will be denoted $Toroid_{\gamma}(P)$, or simply $Toroid_{\gamma}$ when the point P is implied

Similarly, we define $Toroid_{\alpha}(P)$ and $Toroid_{\beta}(P)$

If instead of the arc through P, we use the other arc connecting A and B, on the same circle, the one that does not go through P, then we obtain the *complementary toroid,* $CToroid_{\gamma}(P)$

Similarly for $CToroid_{\alpha}(P)$ and $CToroid_{\beta}(P)$





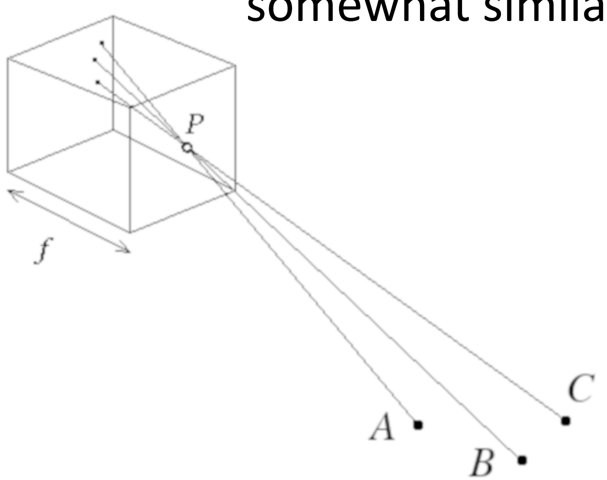
Toroids

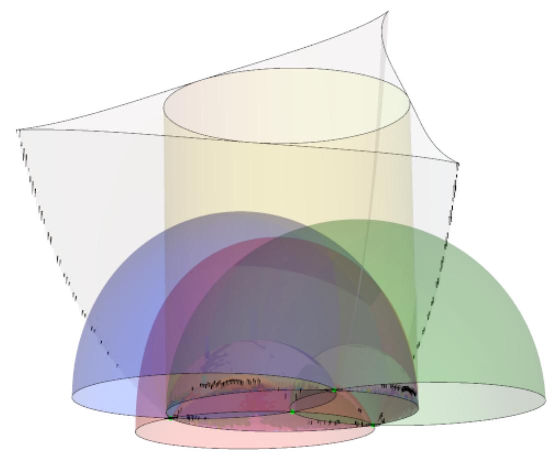
The union of $Toroid_\gamma$(P) and $CToroid_\gamma$(P) is of course the surface obtained by rotating the entire circle ABP about the line AB

We will call this a *double toroid,* and denote it as $DToroid_\gamma$(P)

Similarly for $DToroid_\alpha$(P) and $DToroid_\beta$(P)

We will now see that $Toroid_\alpha$(P), $Toroid_\beta$(P) and $Toroid_\gamma$(P) are closely associated with the concept of "strongly related," and later that $DToroid_\alpha$(P), $DToroid_\beta$(P) and $DToroid_\gamma$(P) are somewhat similarly associated with "weakly related"

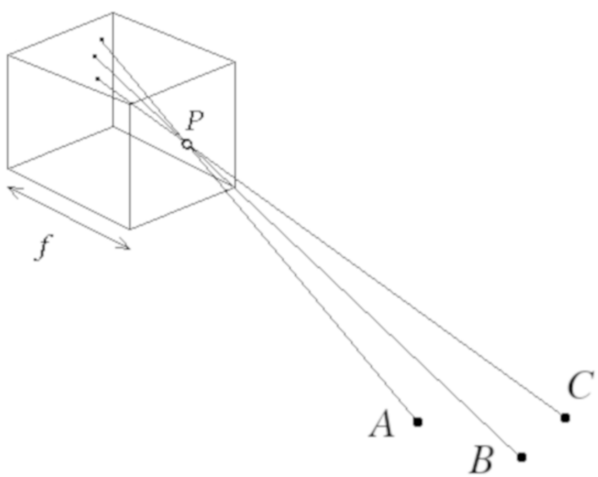


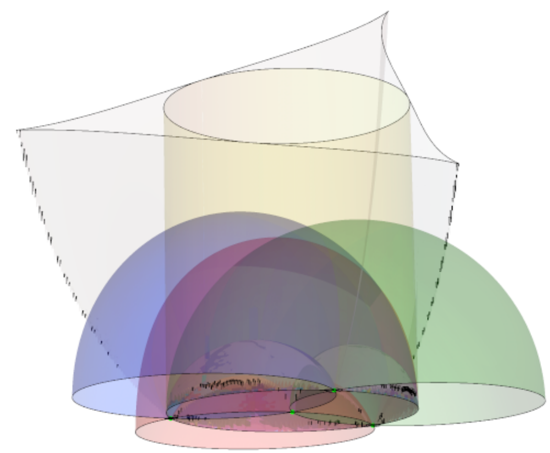


Toroids

The following are equivalent for points P and Q:

- P and Q are strongly related (*i.e.* have the same c_1, c_2 and c_3 values)
- P and Q have the same values of α, β and γ
- Q is in the intersection of $\mathit{Toroid}_\alpha(P)$, $\mathit{Toroid}_\beta(P)$ and $\mathit{Toroid}_\gamma(P)$
- P is in the intersection of $\mathit{Toroid}_\alpha(Q)$, $\mathit{Toroid}_\beta(Q)$ and $\mathit{Toroid}_\gamma(Q)$



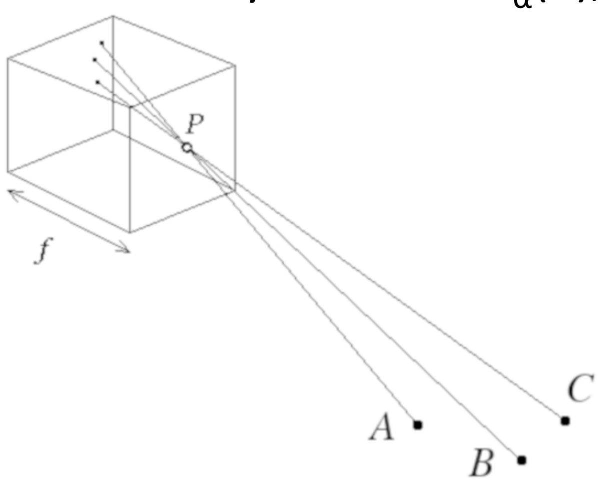


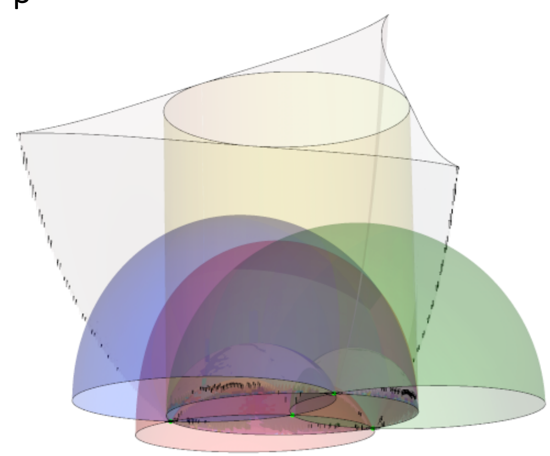
Toroids

Letting γ denote the angle at P between the rays to A and B, it is straightforward to check that the points on $CToroid_\gamma(P)$ are the points at which the angle between the rays to A and B is $\pi - \gamma$ (whose cosine is of course $-\cos\gamma$)

The points on $DToroid_\gamma(P)$ are the points that have the same value of c_3^2 (*i.e.* same value of $|c_3|$) as P

Similarly for $CToroid_\alpha(P)$, $CToroid_\beta(P)$, $DToroid_\alpha(P)$ and $DToroid_\beta(P)$





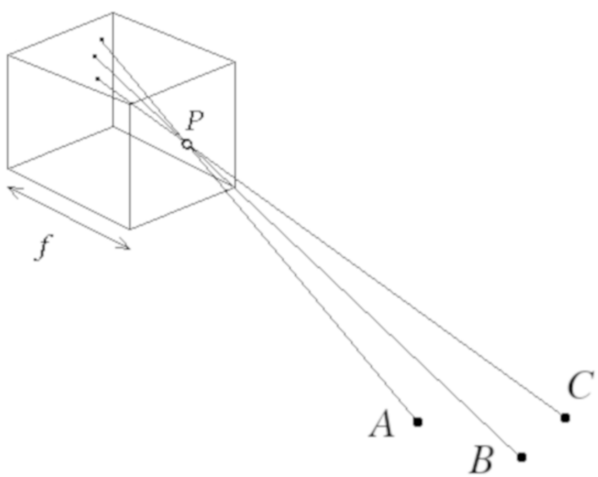
Toroids

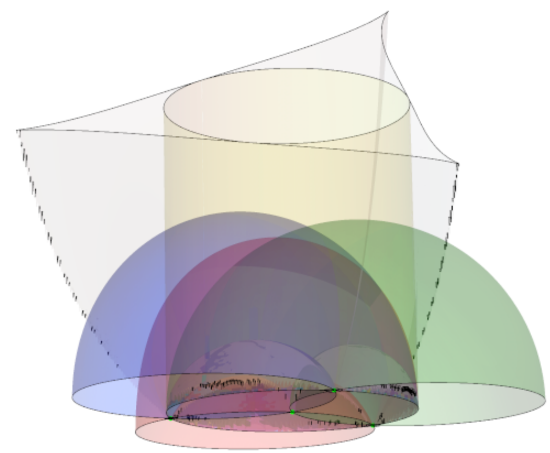
The intersection of $DToroid_\alpha(P)$, $DToroid_\beta(P)$ and $DToroid_\gamma(P)$ thus consists of points that have the same values of c_1^2, c_2^2 and c_3^2 as P

There are only finitely many such points

However, in order for a point to be weakly related to P, it must also have the same value of $c_1 c_2 c_3$ as P

This means it must be on an odd number of $Toroid_\alpha(P)$, $Toroid_\beta(P)$ and $Toroid_\gamma(P)$



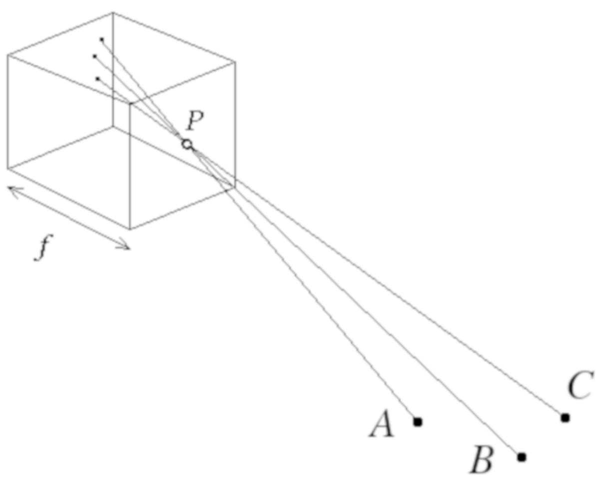


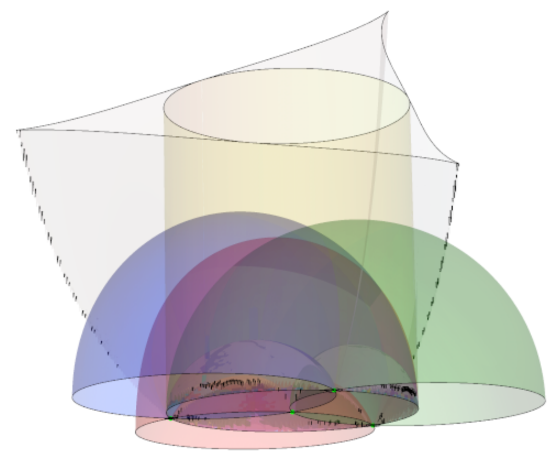
Toroids

Toroids and double toroids of particular interest occur when P is A, B or C

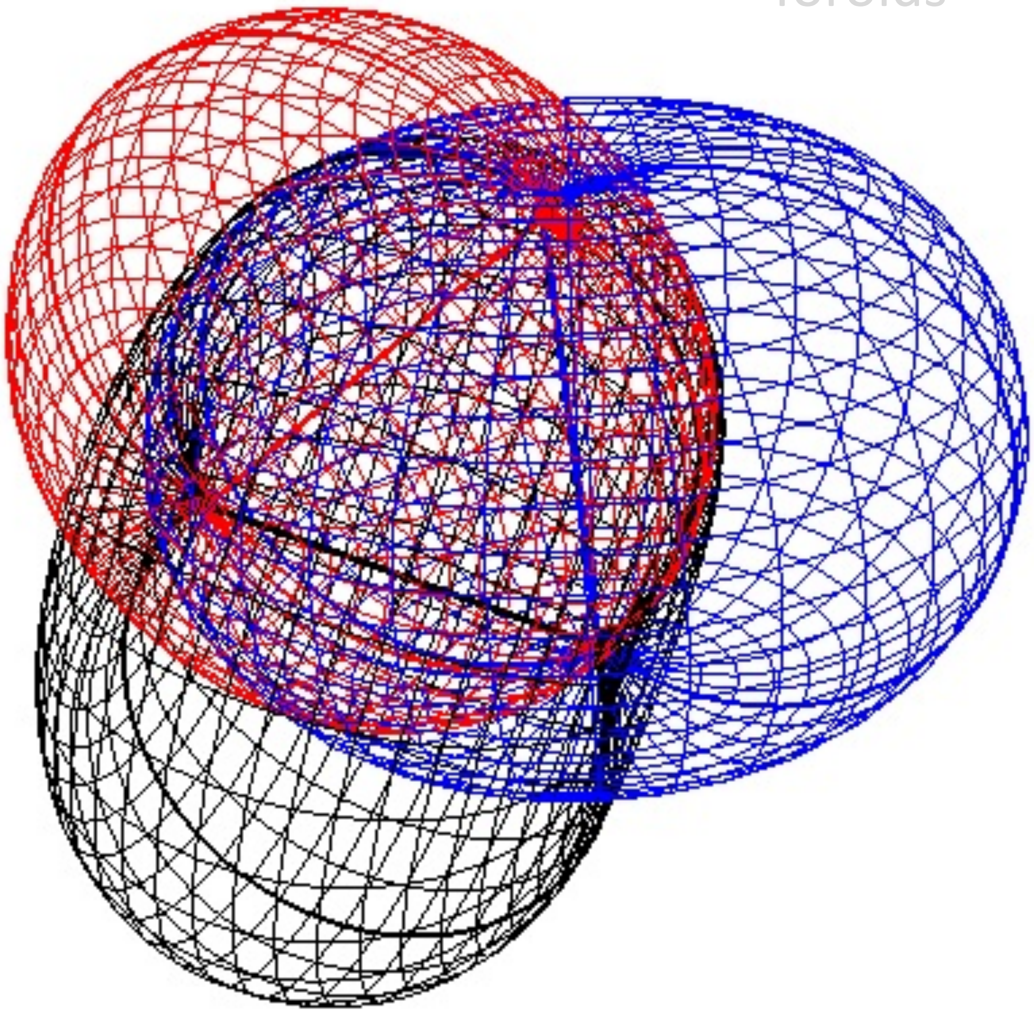
Let us define $\mathit{Toroid}_A = \mathit{Toroid}_\alpha(A)$, $\mathit{Toroid}_B = \mathit{Toroid}_\beta(B)$ and $\mathit{Toroid}_C = \mathit{Toroid}_\gamma(C)$

Likewise, define $\mathit{CToroid}_A = \mathit{CToroid}_\alpha(A)$, $\mathit{CToroid}_B = \mathit{CToroid}_\beta(B)$, $\mathit{CToroid}_C = \mathit{CToroid}_\gamma(C)$, $\mathit{DToroid}_A = \mathit{DToroid}_\alpha(A)$, $\mathit{DToroid}_B = \mathit{DToroid}_\beta(B)$ and $\mathit{DToroid}_C = \mathit{DToroid}_\gamma(C)$





Toroids

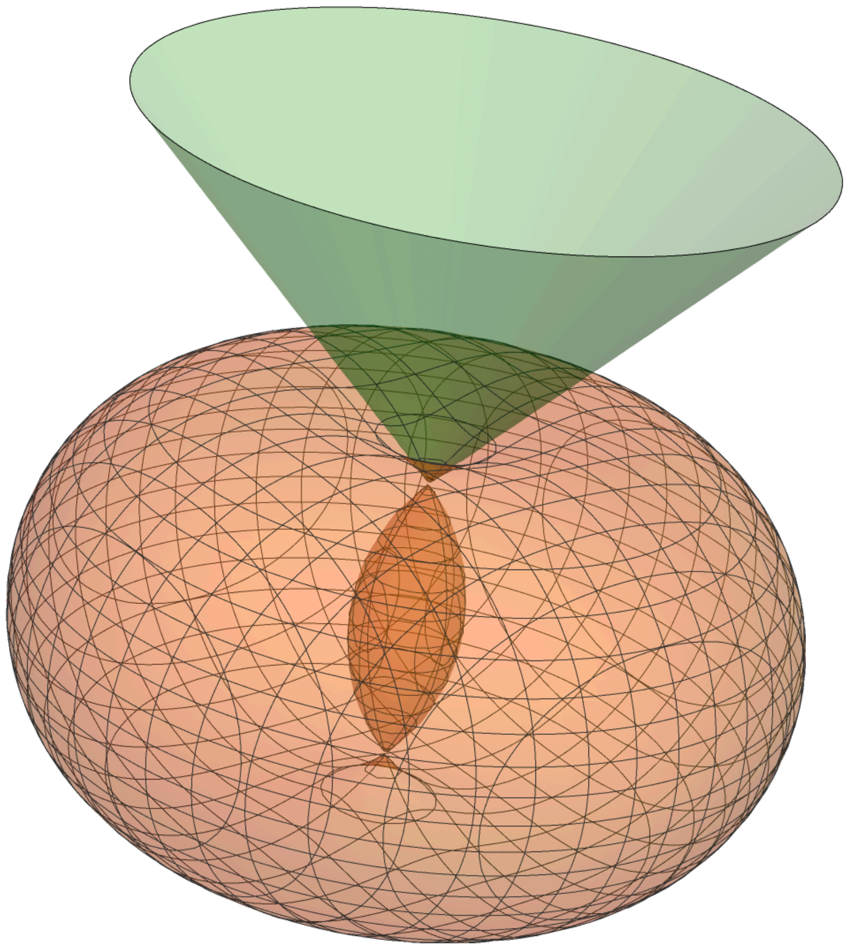


The three special toriods:
***Toroid*$_A$, *Toroid*$_B$ and *Toroid*$_C$**

When the camera's pinhole is outside of the union of *Toroid*$_A$, *Toroid*$_B$ and *Toroid*$_C$, no strict weakly related solution points exists; that is, all of its weakly related solution points are strongly related.

When the camera's pinhole passes through one of the three special toroids, *Toroid*$_A$, *Toroid*$_B$ or *Toroid*$_C$, one of its weakly related solution points passes through the corresponding vertex, and changes how it is related to the camera's pinhole

Toroids



A double toroid, and the tangent cone for its outer toroid at one of the singularities

Notice the inner toroid inside the outer toroid

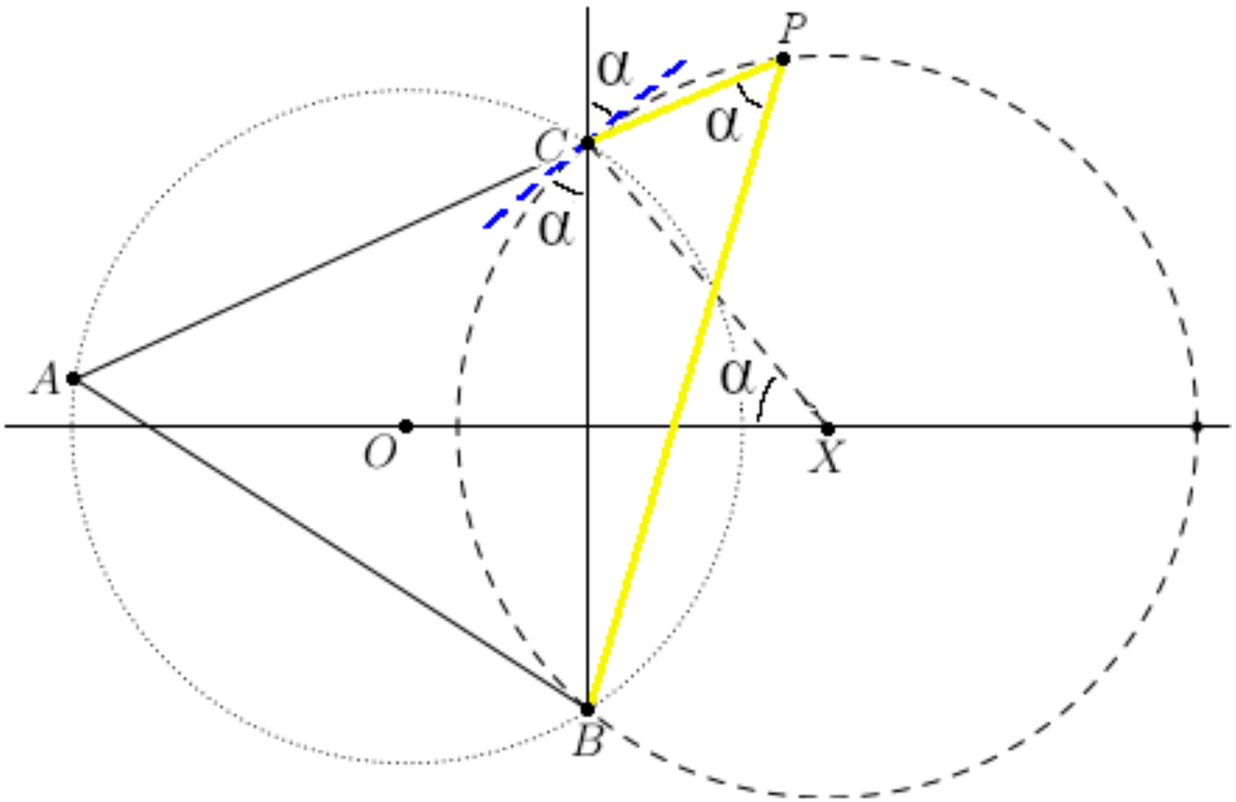
The cone's apex is of course one of the two singular points of the double toroid

Each half-line on the cone, from its apex, is tangent to the outer toroid

Cute fact: The angle, at any point on the outer toroid, between the rays from it to the toroid's two singularities, is equal to the angle, at the cone's apex, between the cone's half-axis and any of its half-lines from the apex

This can be reasoned using the next figure

Toroids



(Altered version of Danny Maienschein's figure)

We are looking here at half of the intersection of a double toroid *DToroid*$_\alpha$ and the xy-plane, specifically the dashed circle in the figure

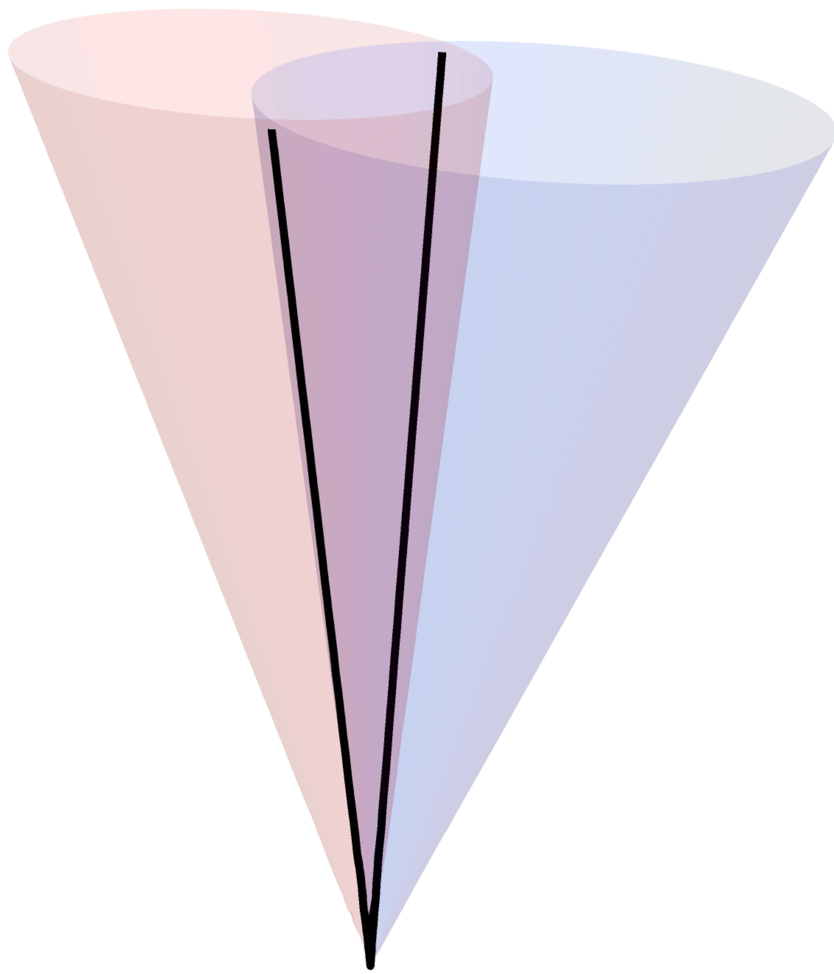
By the Inscribed Angle Theorem, the ***viewing angle*** α for *Toroid*$_\alpha$, at P for instance, is congruent to the angle at the center X of the dashed circle, between the segments XC and XO, where O is the circumcenter of triangle ΔABC

By other theorems in planar geometry (with a little calculus?), this angle is also congruent to the other two angles in the figure labeled "α"

But one of these is the angle at the apex of the tangent cone, between any of its half-lines from the apex, and its half-axis

The angle labeled α at the point P in the figure is one of the three classical ***angular coordinates*** for P (investigated by Maienschein and Rieck in connection with other planar and algebraic geometry concepts)

Toroids



When they intersect, two circular cones with the same apex typically intersect in a pair of half-lines whose ends are at the apex

Of course, when the two cones are tangent to each other, they intersect in just one such half-line

This is of use for P3P when dealing with the tangent cones of the toroids that occur in P3P

Such cone intersections tell us something about the intersections of toroids that have a common singularity

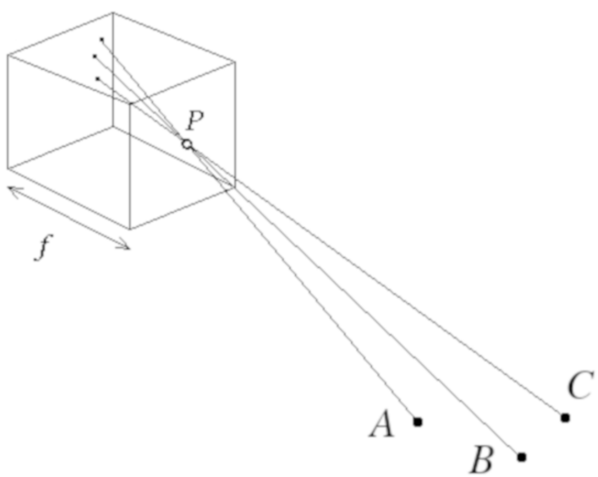
Toroids

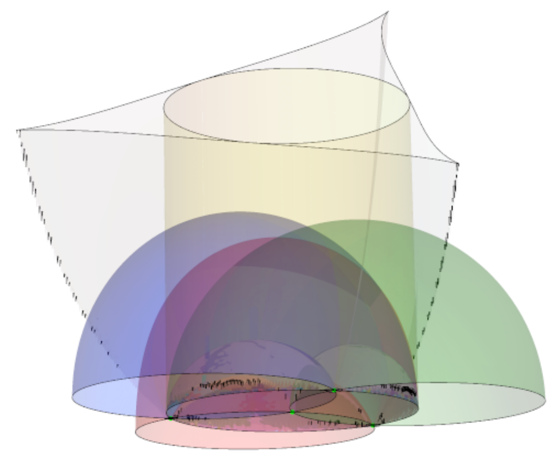
Lemma

If $Toroid_C$ uses the longer of the two circular arcs connecting A and B, then none of it is inside the unit sphere, but if instead it uses the shorter of the two circular arcs connecting A and B, then none of it is outside the unit sphere

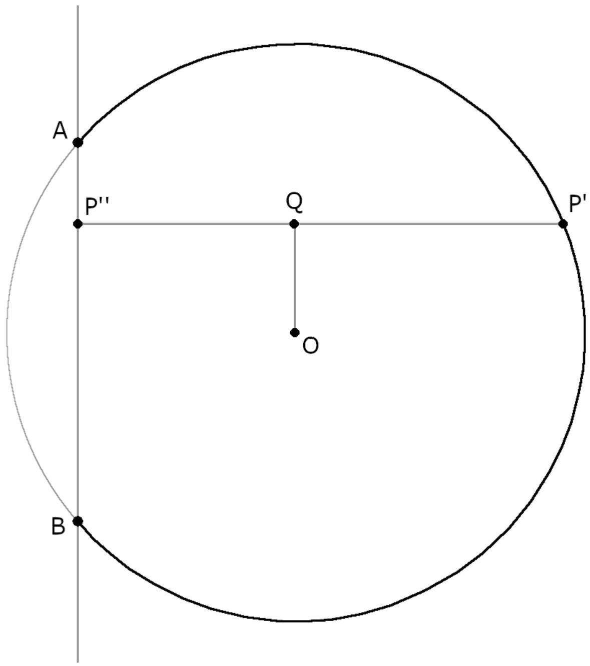
Similarly for $Toroid_A$ and $Toroid_B$

The next three slides outline a proof of this lemma





Toroids



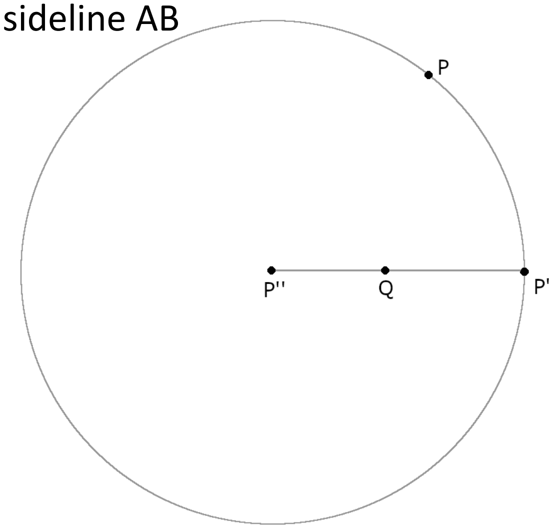
Assume first that the longer arc is used to generate the toroid

Any point P on the toroid is gotten by rotating a point P’ on the arc

Consider the line through the circumcenter O that is also parallel to the sideline AB

Assume first that P’ is not on the side of this line that contains A and B

Let P’’ be the orthogonal projection P’ onto the sideline AB



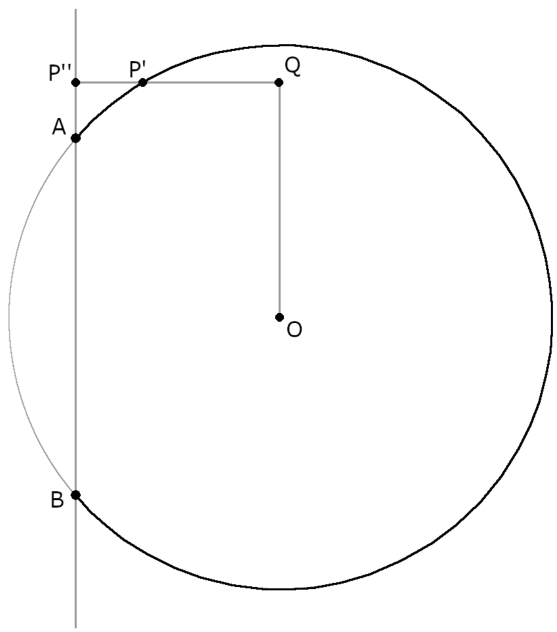
Let Q be the orthogonal projection of O onto the line P’P’’

Looking at the “side view” on the right, it is clear that d(Q, P) ≥ d(Q, P’)

So $d(O, P)^2 = d(O, Q)^2 + d(Q, P)^2 \geq d(O, Q)^2 + d(Q, P')^2 = d(O, P')^2 = 1$

So d(O, P) ≥ 1, and we see that the toroid never goes inside the unit sphere

Toroids

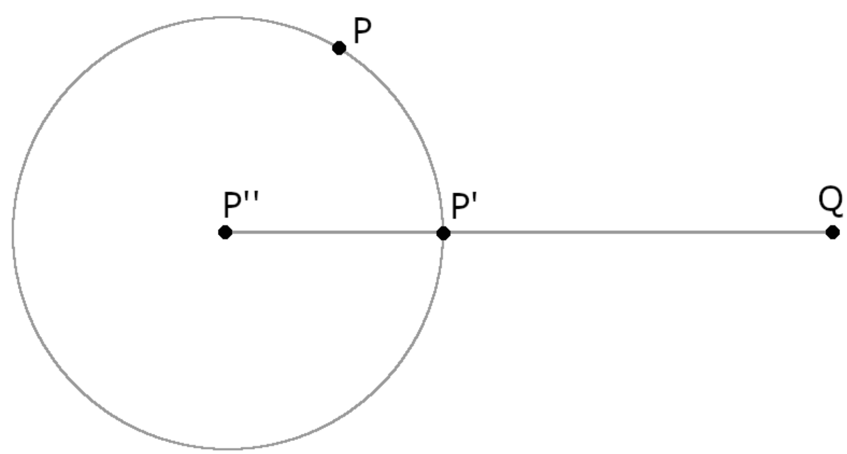


Let us now alter the previous situation only by assuming that A, B and P’ are on the same side of the line through O that is parallel to the sideline AB

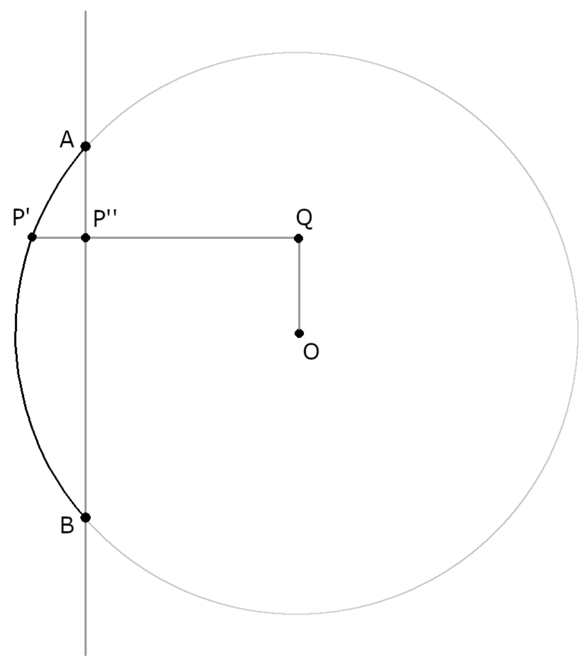
This alters the “side view” as seen in the following figure

Nevertheless, we still have $d(Q, P) \geq d(Q, P')$

We are the lead to the same conclusion as before, in exactly the same way `



Toroids

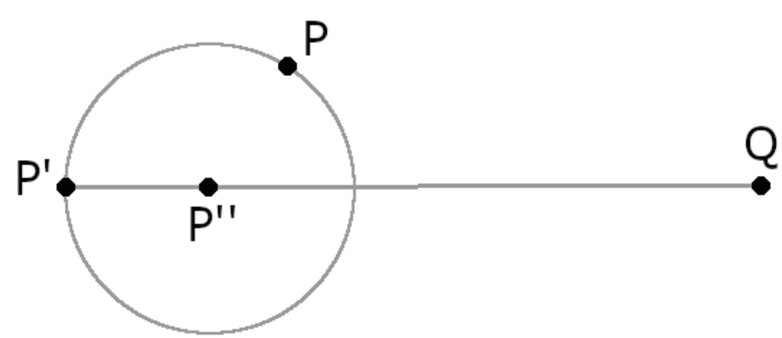


Now, if we instead start over using the shorter arc instead of the longer arc, things completely change

The “side view” now reveals that $d(Q, P) \leq d(Q, P')$

We are then easily lead to conclude that $d(O, P) \leq 1$

Thus we see that the toroid never goes outside the unit sphere



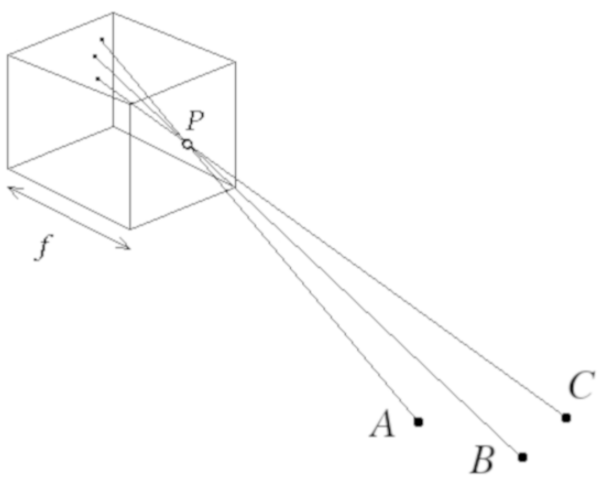
Toroids

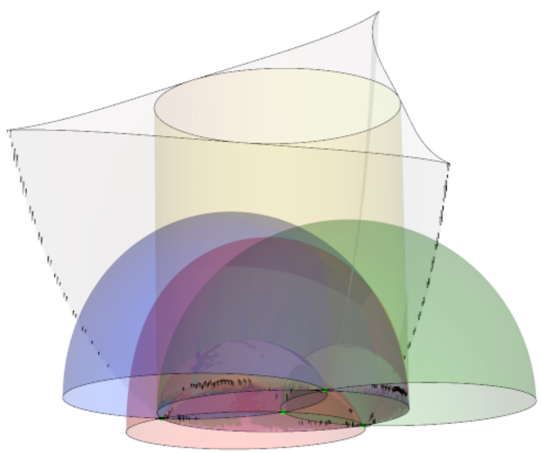
<u>Definitions</u> (Categories of strict weakly related points)

α-strict weakly related points: same sign for c_1*, but not* c_2 *and* c_3

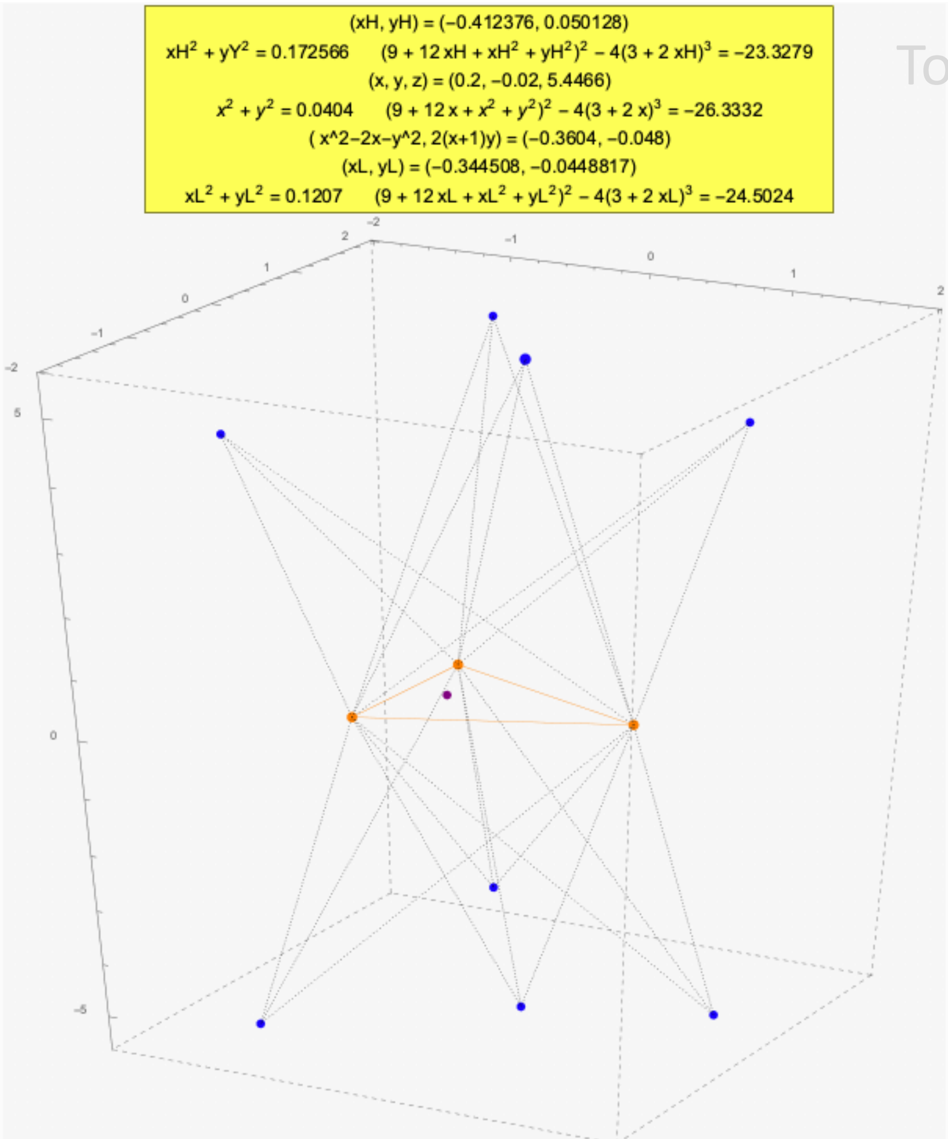
β-strict weakly related points: same sign for c_2*, but not* c_3 *and* c_1

γ-strict weakly related points: same sign for c_3*, but not* c_1 *and* c_2





Toroids



(Wait for animation)

Here we see the three different types of strict weakly related points (relative to the camera's pinhole), using three different colors (cyan, green and gray)

When passing through a control point, a strongly related point will switch to a strict weakly related point of the type associated with that control point, and vice versa

When a strict weakly related point passes through a control point, and has a type not associated with that control point, it switches to the other type not associated with that control point

For instance, passing through the control point on the left switches a moving point between blue (strongly related) and cyan (weakly related), and also between gray and green (both weakly related)

Toroids

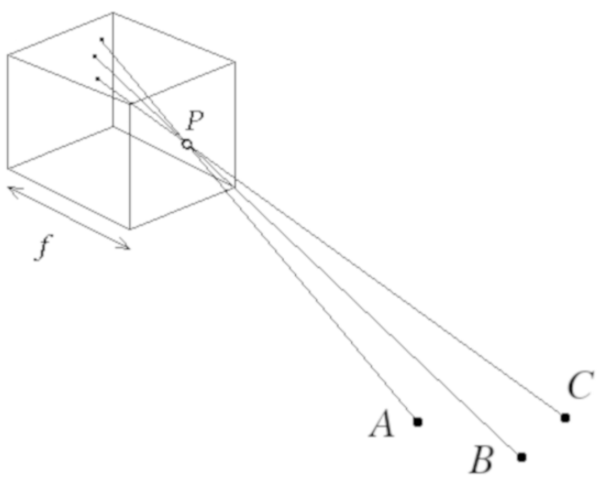
<u>Definitions</u> (THE REST OF THIS SECTION IS UNDER DEVELOPMENT)

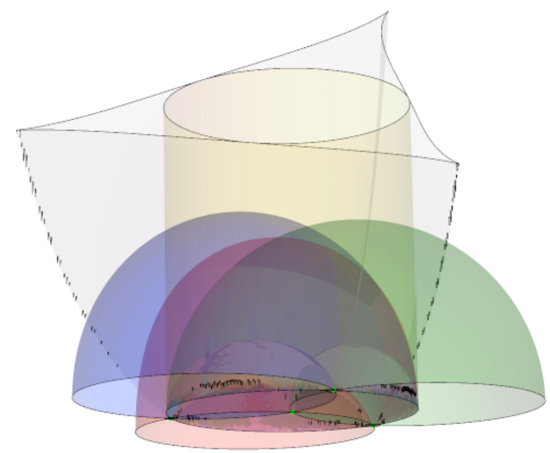
Cone$_{A,\ \beta}$: the tangent cone at A, with line AC as axis, and with viewing angle β

Ray$_{A,\ \beta,\ \gamma}$: one of the rays in the intersection of *Cone*$_{A,\ \beta}$ and *Cone*$_{A,\ \gamma}$

Normal$_{A,\ \Delta ABC}$: the normal of the circumcircle of triangle ΔABC, at point A

Angle$_{A,\ \beta,\ \gamma}$: the included angle of *Ray*$_{A,\ \beta,\ \gamma}$ with *Normal*$_{A,\ \Delta ABC}$



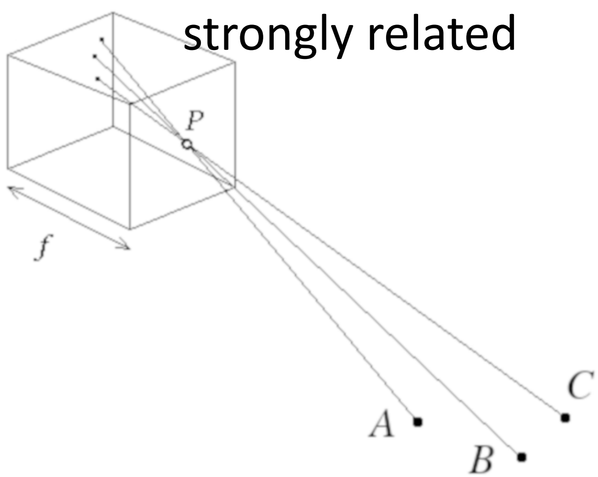


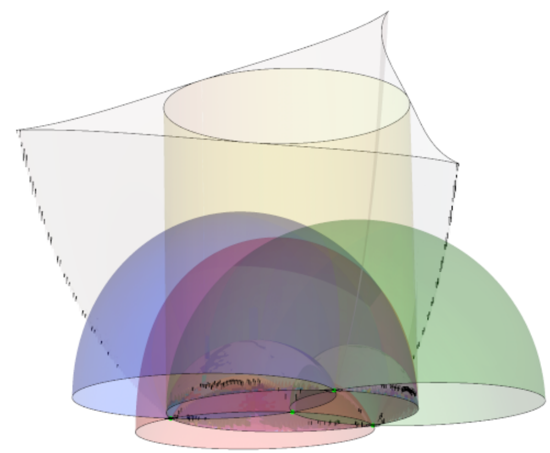
Toroids

The β and γ values of a moving point P, when moving into Toroid_A, are such that $\mathit{Angle}_{A,\beta,\gamma}$ determines exactly how the related point passing through A changes

If $\mathit{Angle}_{A,\beta,\gamma} < \pi/2$, the related solution change from strongly related to strict weakly related

If $\mathit{Angle}_{A,\beta,\gamma} > \pi/2$, the related solution change from strict weakly related to strongly related

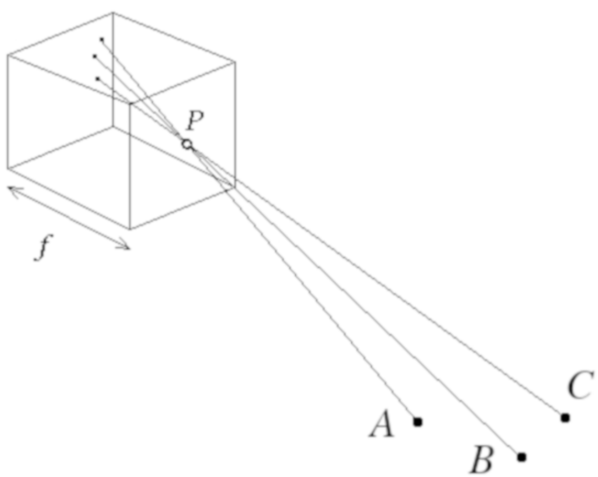


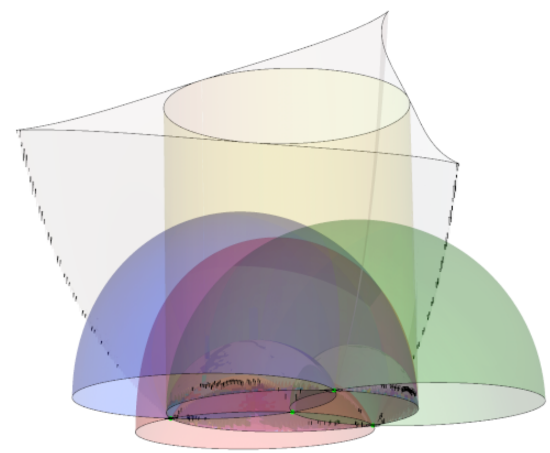


The Limiting Case of P3P

In general, given a camera pinhole position (x_0, y_0, z_0) in space, it is quite difficult to describe where its related points are located

However, when we consider the *limiting case* for which $|z_0|$ becomes large without bound, but x_0 and y_0 stay bounded, the situation can be very eligently described



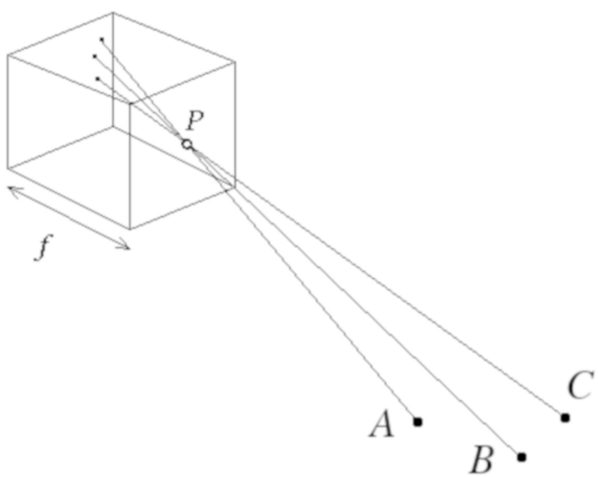


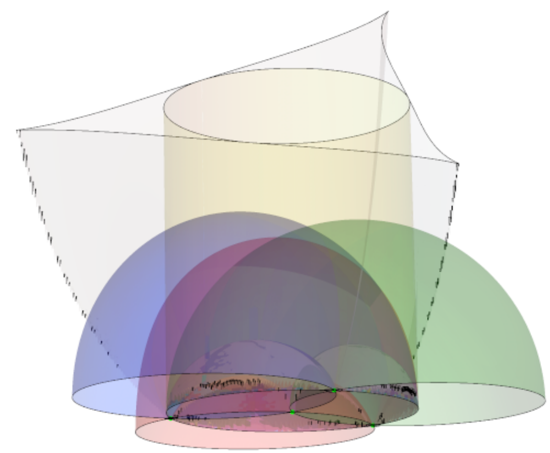
The Limiting Case of P3P

As the camera pinhole's z-coordinate goes to infinity, its weakly related points in the upper half-space are strongly related, and have z-coordinates that also go to infinity

The projections, in the limit as $|z_0|$ goes to infinity, of all these points onto the xy-plane are at the intersections of the following two rectangular hyperbolas:

$$y^2 - (x-1)^2 = y_0^2 - (x_0-1)^2 \quad \text{and} \quad 2(x+1)y = 2(x_0+1)y_0$$



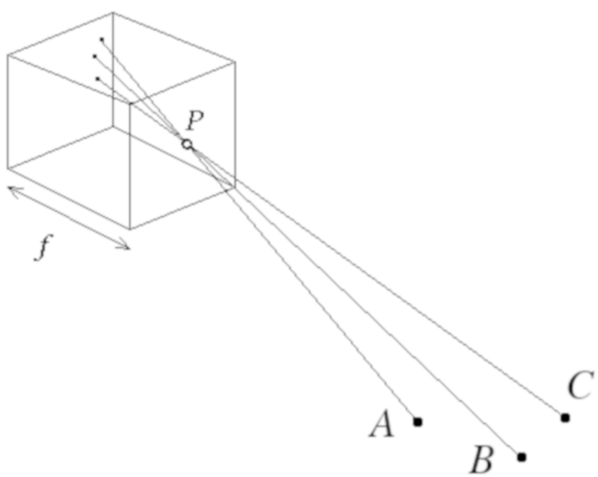


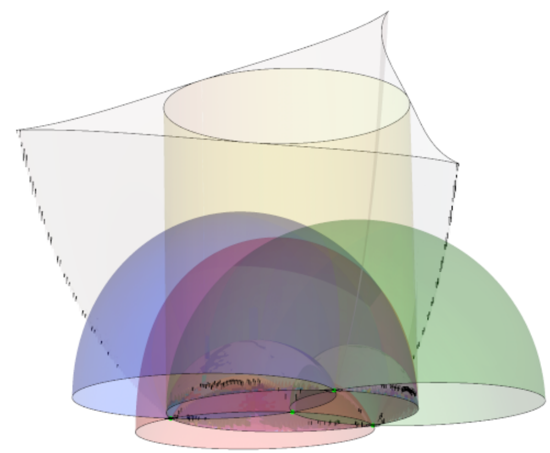
The Limiting Case of P3P

Generally speaking, these two hyperbolas intersect in either two or four points

Whether there are two or four intersection points depends on the sign of the following quantity: $(9 + 12\, x_0 + x_0^2 + y_0^2)^2 - 4(3+2x_0)^3$

The curve given by the equation $(9 + 12\, x + x^2 + y^2)^2 = 4(3+2x)^3$ is well-known, and is called the (standard) *deltoid curve*



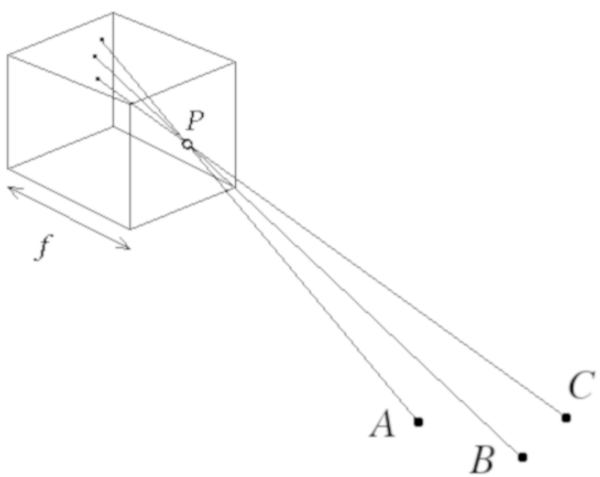


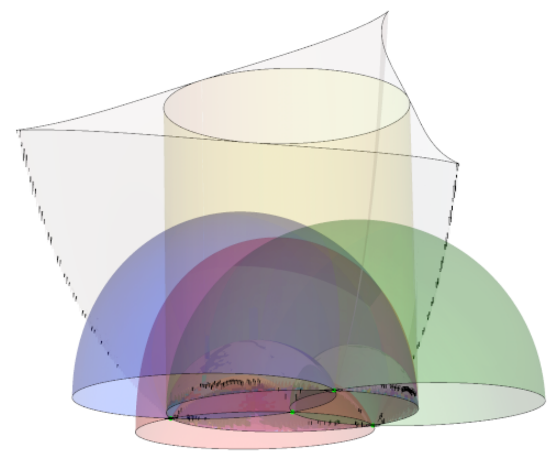
The Limiting Case of P3P

When the two rectangular hyperbolas intersect in two points, these are outside the deltoid curve

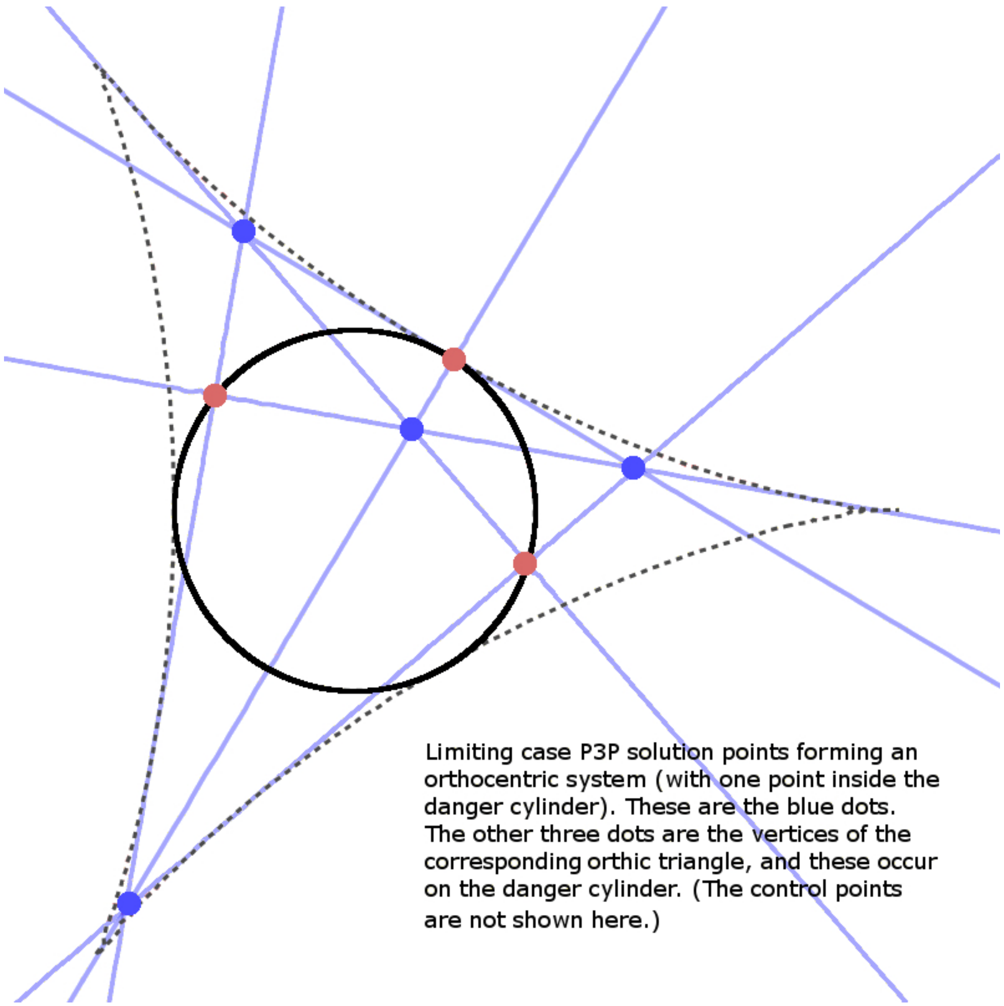
When the two rectangular hyperbolas intersect in four points, these are inside the deltoid curve, and one is inside the unit circle

These four points form an *orthocentric system,* meaning that the line through any two of the four points is perpendicular to the line through the other two points





The Limiting Case of P3P



Limiting case P3P solution points forming an orthocentric system (with one point inside the danger cylinder). These are the blue dots. The other three dots are the vertices of the corresponding orthic triangle, and these occur on the danger cylinder. (The control points are not shown here.)

Here the camera is vertically very far away from the control points, but *not* horizonally far away

Since the camera's pinhole is inside the deltoid curve, so are all of its (strongly) related points (blue dots)

They form an orthocentric system

One blue dot is inside the unit circle

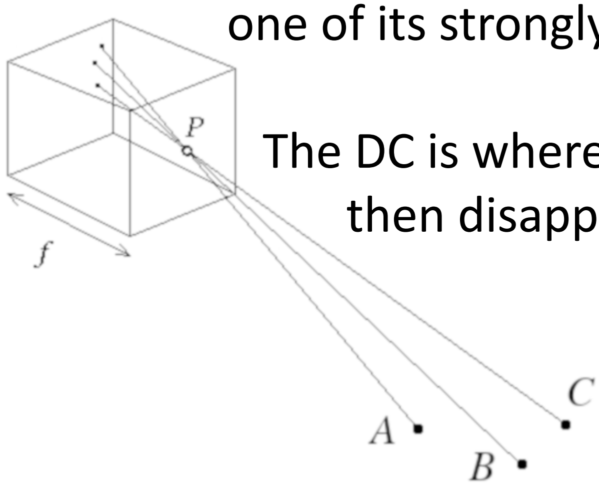
(Ignore the reddish dots)

The Danger Cylinder and its Companion Surface

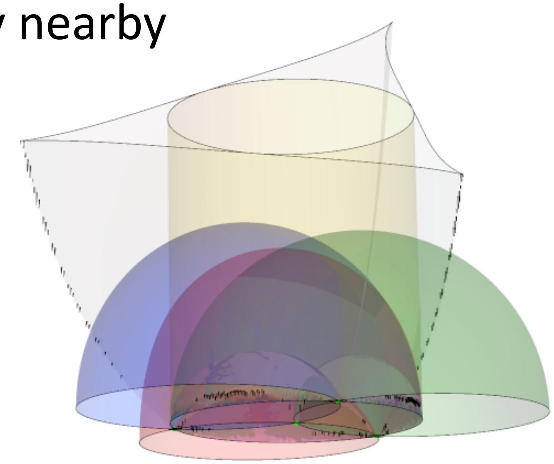
We will henceforth regard the control points as the vertices of a triangle, and make routine mention of its circumcircle, orthocenter, interior angles, and so forth

It has long been recognized that the circular cylinder obtained by extending the circumcircle perpendicular to the plane containing the control points, plays a special role in the P3P problem, and it was long ago named the *danger cylinder (DC)*

If the camera's pinhole is infinitesimally close to a point on the danger cylinder, then one of its strongly related points will be infinitesimally nearby



The DC is where related points come together and then disappear (due to complex solutions)

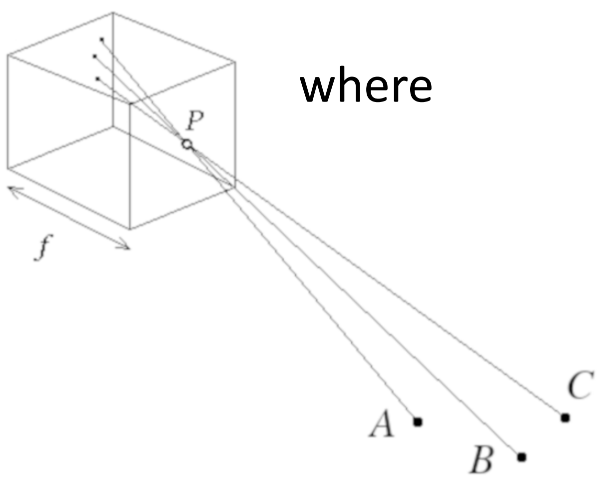


The Danger Cylinder and its Companion Surface

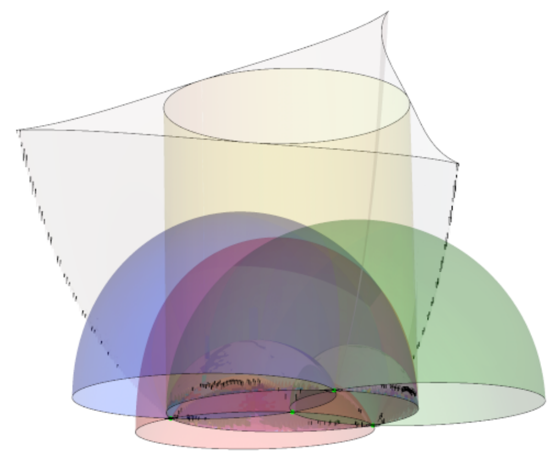
We can say that if Grunert's system (with given parameters) has a solution point on the danger cylinder, then this is actually a *repeated solution point,* and all repeated solution points are thus

Whether or not Grunert's system has a repeated solution point (on the danger cylinder) is determined by whether or not

$$(9 + 12\,x_L + x_L^2 + y_L^2)^2 = 4\,(3+2x_L)^3$$



where



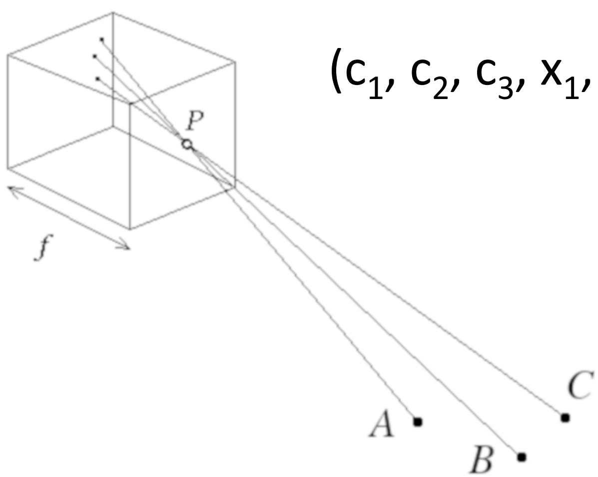
The Danger Cylinder and its Companion Surface

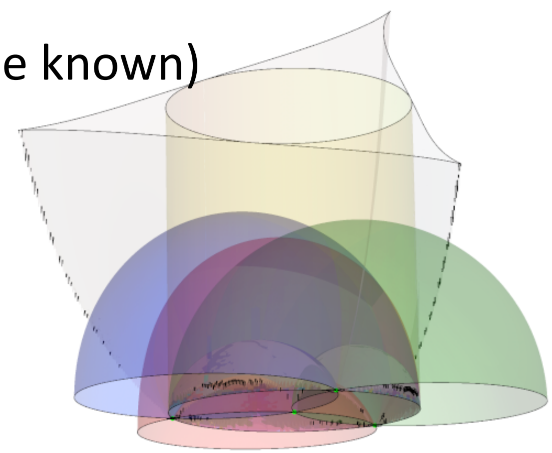
$$x_L = [\,-3 + 2(x_1^2+x_2^2+x_3^2) + (1-2x_1-2x_1^2)\,c_1^2 + (1-2x_2-2x_2^2)\,c_2^2 + (1-2x_3-2x_3^2)\,c_3^2 + 2\,(x_1+x_2+x_3)\,c_1c_2c_3\,]\;/\;[\,1 - c_1^2 - c_2^2 - c_3^2 + 2c_1c_2c_3\,]$$

and

$$y_L = [\,2(x_1y_1+ x_2y_2+ x_3y_3) + 2(1-x_1)y_1\,c_1^2 + 2(1-x_2)y_2\,c_2^2 + 2(1-x_3)y_3\,c_3^2 - 2\,(y_1+y_2+y_3)\,c_1c_2c_3\,]\;/\;[\,1 - c_1^2 - c_2^2 - c_3^2 + 2c_1c_2c_3\,]$$

(c_1, c_2, c_3, x_1, x_2, x_3, y_1, y_2, y_3 are all presumed to be known)



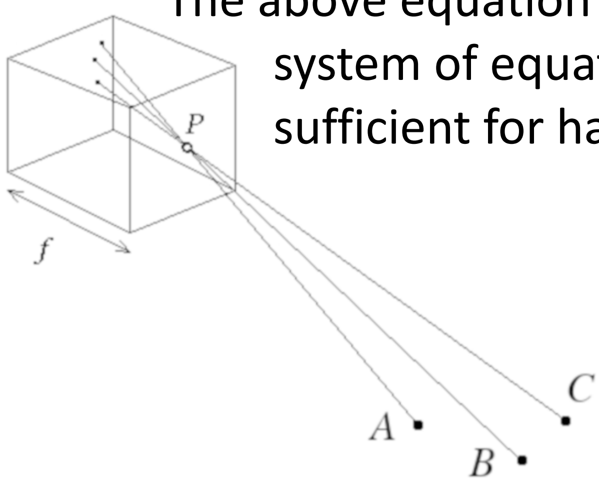


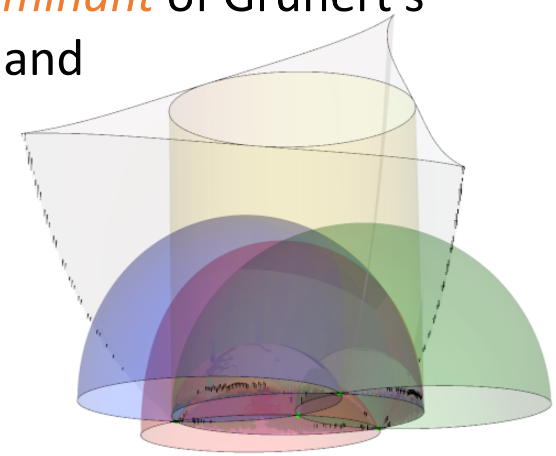
The Danger Cylinder and its Companion Surface

To discover which points in space are solutions to Grunert's system, when this system has a repeated solution, one can express c_1, c_2, c_3, x_L and y_L in terms of x, y and z, and then rely on algebraic manipulation software

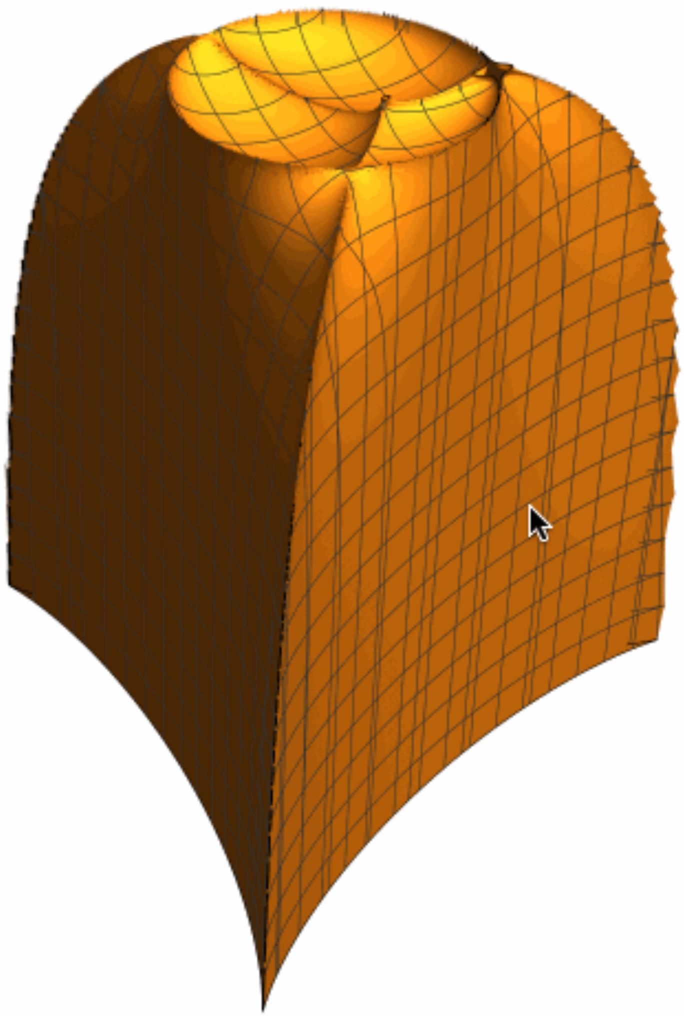
It turns out that all such points lie either on the danger cylinder or else on a surface that we call the *companion surface to the danger cylinder (CSDC)* (though it has also been previously called the "deltoidal surface" by one of us)

The above equation could reasonably be called the *discriminant* of Grunert's system of equations since its vanishing is necessary and sufficient for having a repeated solution





The Danger Cylinder and its Companion Surface



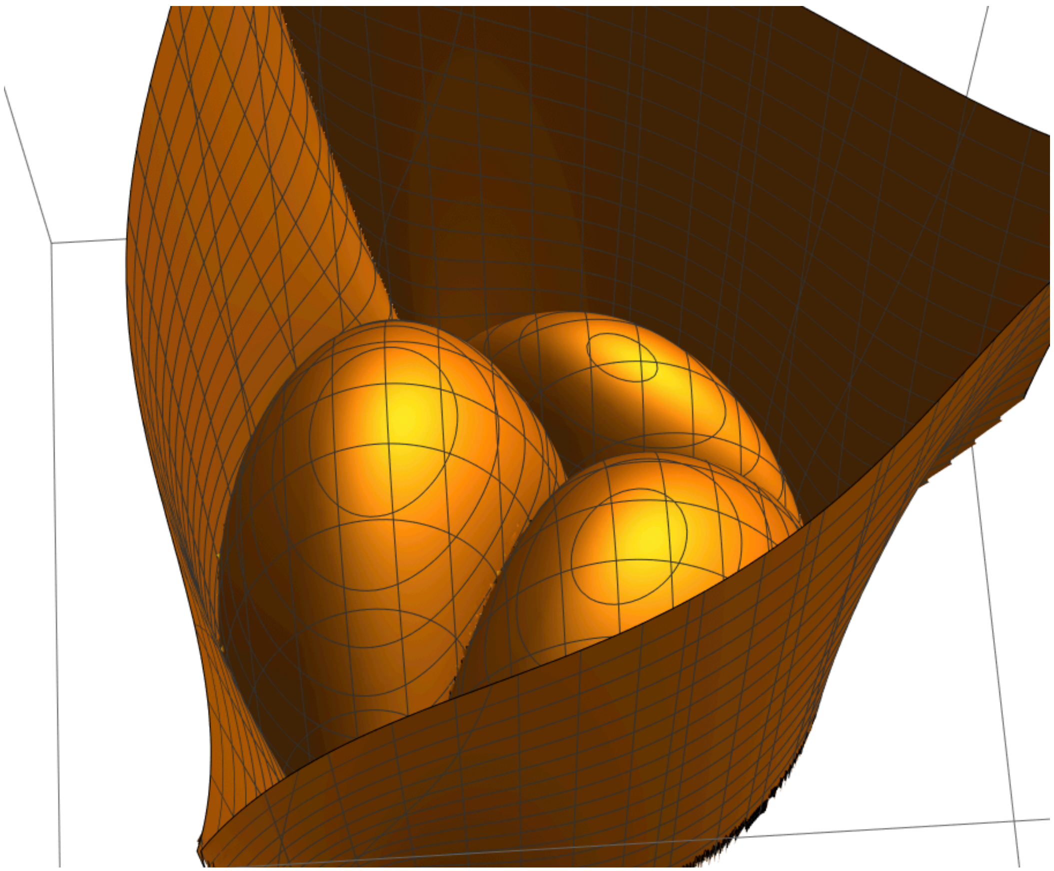
(Wait for animation)

Half of the *CSDC,* in the case where the triangle ΔABC is acute, and so its orthocenter is inside its circumcircle

The other half is the reflection of this about the xy-plane

The vertices (A, B, C) can be seen here, as can the circumcircle, and inside this, the orthocenter H

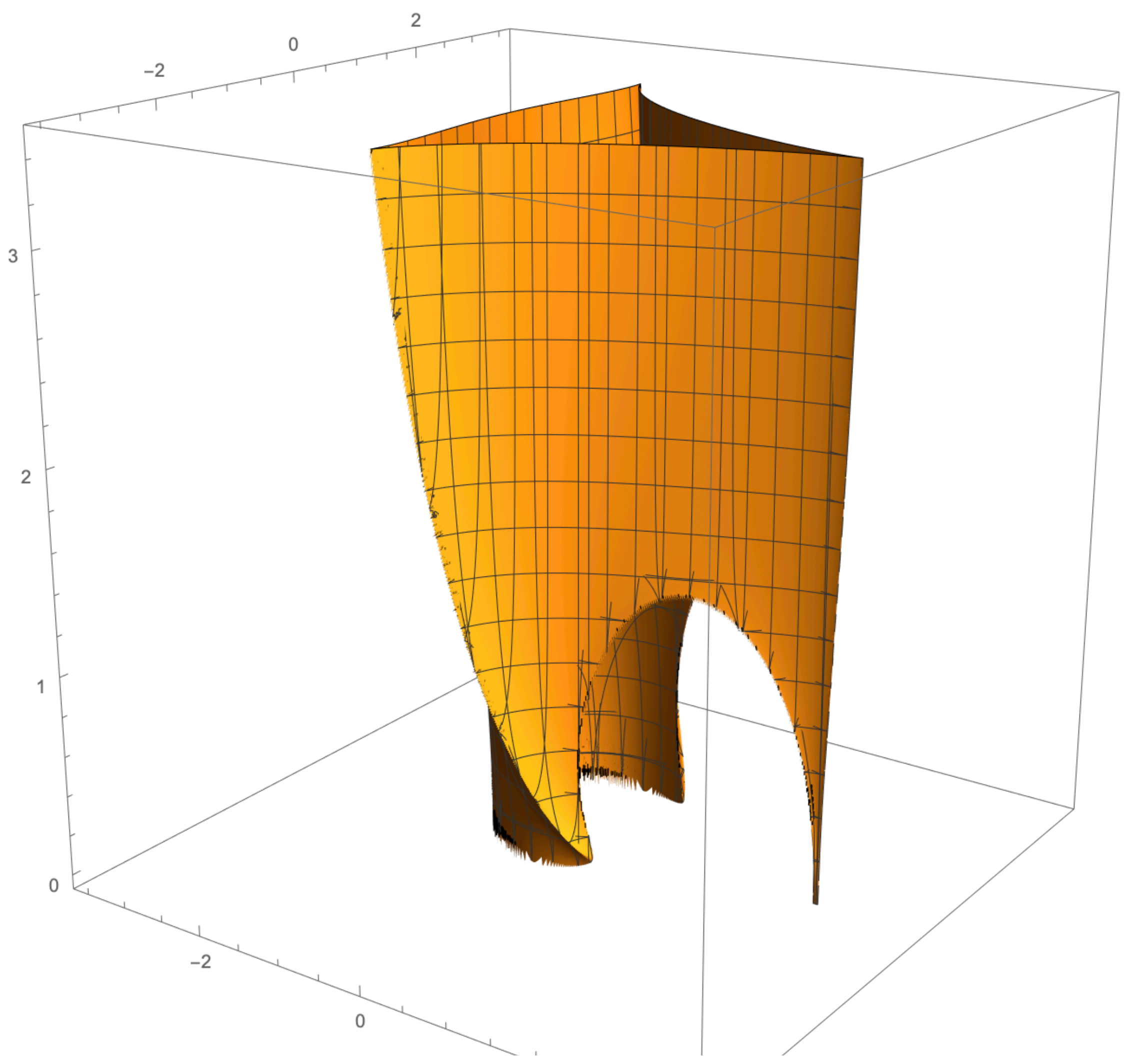
(Ignore the gap near a vertex, which is a rendering issue)



The three *mounds* inside the *CSDC*

When both halves of the *CSDC* are put together, there is a 3D region between a mound and its reflection about the xy-plane, which we call a *chamber*

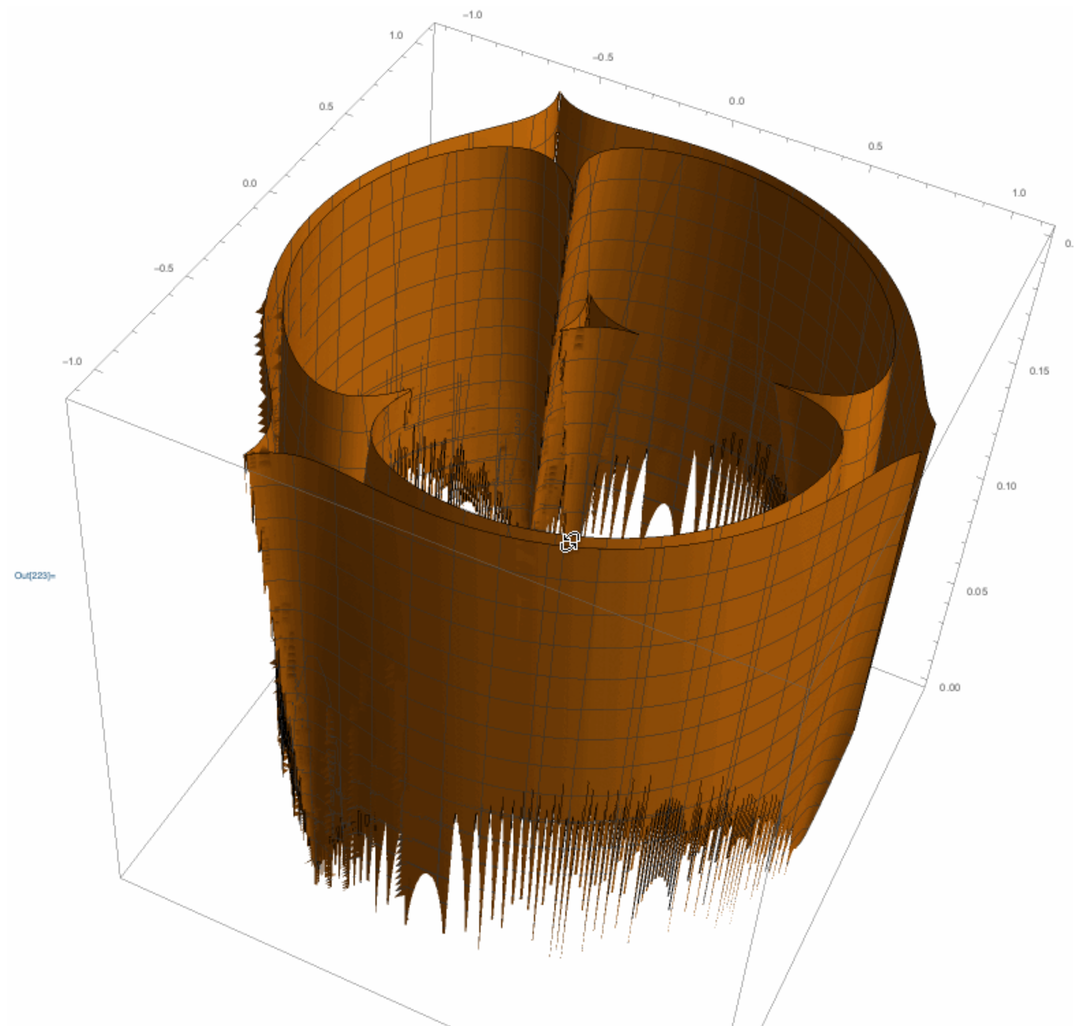
The Danger Cylinder and its Companion Surface



When the triangle ΔABC is obtuse, and so its orthocenter is outside the circumcircle, the *CSDC* looks quite different!

This case is harder to analyze, and has been largely ignored by us

The Danger Cylinder and its Companion Surface



(Wait for animation)

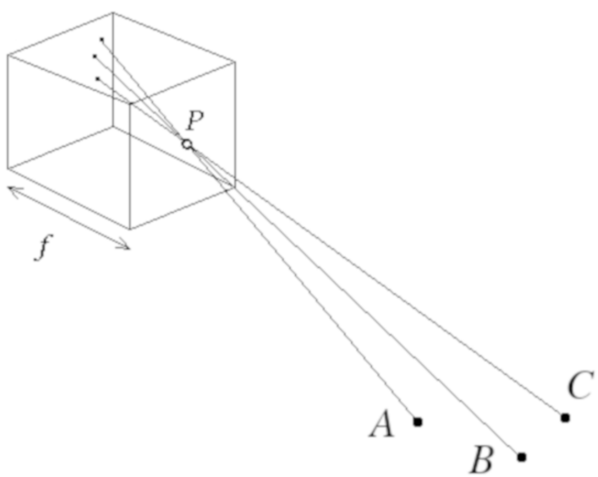
Here is what the *CSDC* looks like close to the xy-plane (the plane containing A, B and C), in the case where the triangle ΔABC is equilateral

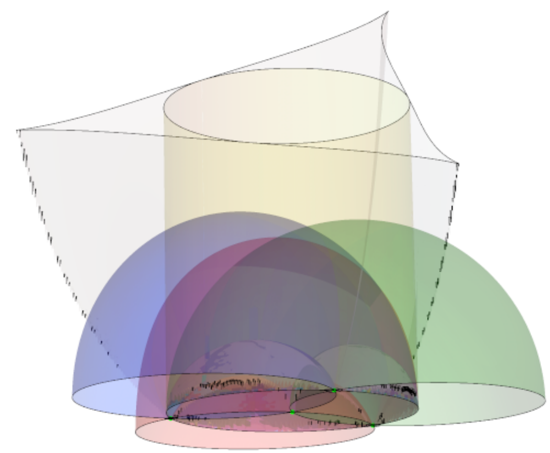
(Ignore the terrible rendering issue very close to the xy-plane)

The Danger Cylinder and its Companion Surface

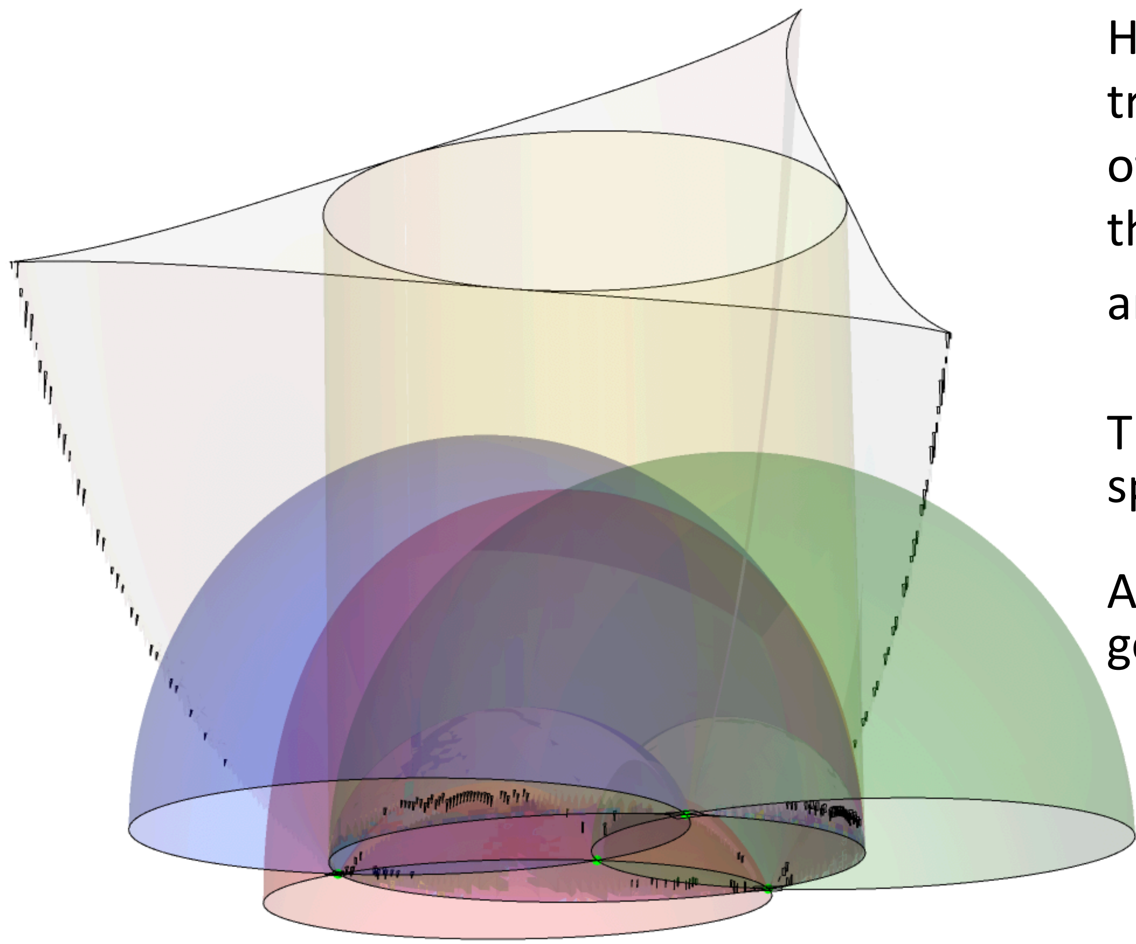
While the equation for the *CSDC*, in its "raw form," is very complicated, it can be rather nicely expresses as a twelfth degree equation in x, y and z with fairly simple coefficients expressed in terms of the coordinates (x_H, y_H) of the orthocenter H of the control points triangle ΔABC

By the way, it turns out that $x_H = x_1 + x_2 + x_3$ and $y_H = y_1 + y_2 + y_3$





Intersecting Surfaces

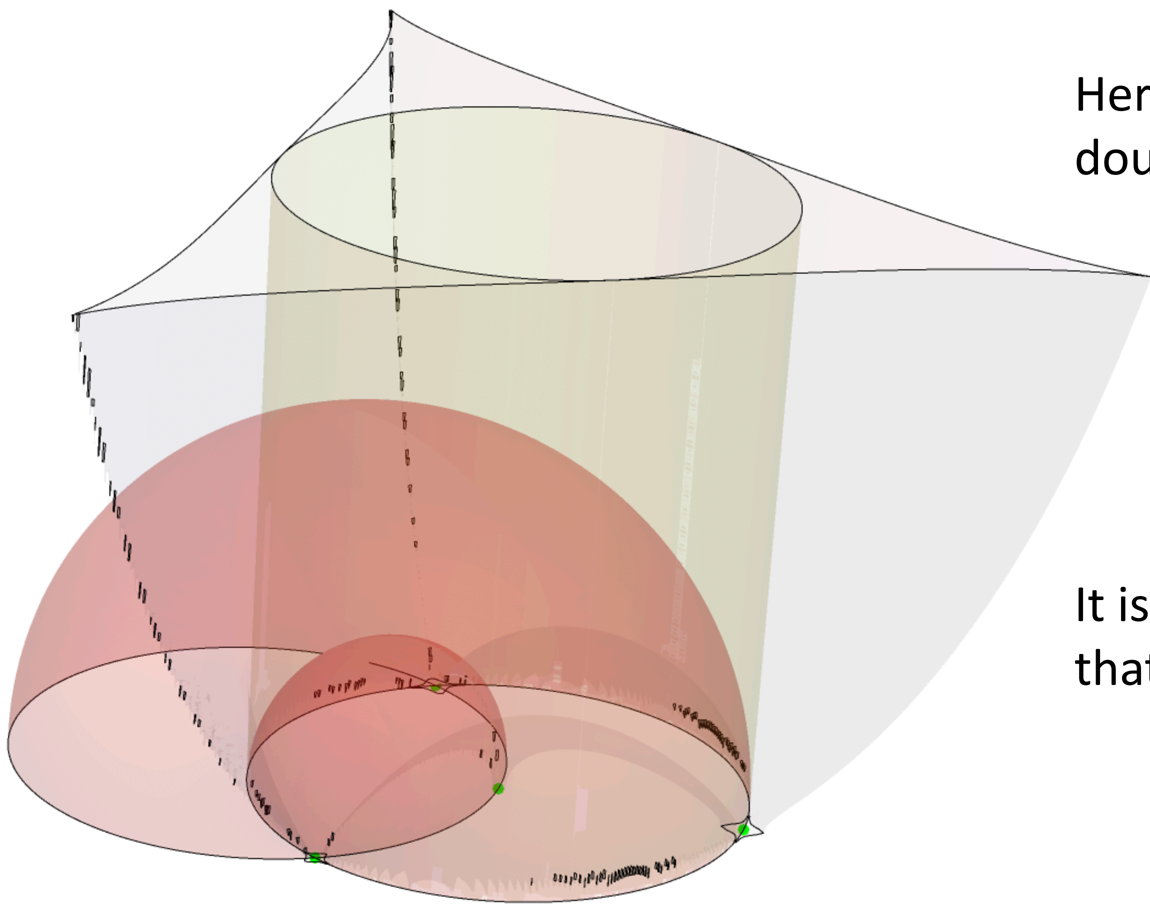


Here is an image, for the equilateral triangle case, that shows five surfaces of importance to the P3P problem: the *DC*, the *CSDC*, $DToroid_A$, $DToroid_B$ and $DToroid_C$

This figure only shows the upper half-space ($z \geq 0$)

Also, we are interested in more general acute triangles, going forward

Intersecting Surfaces



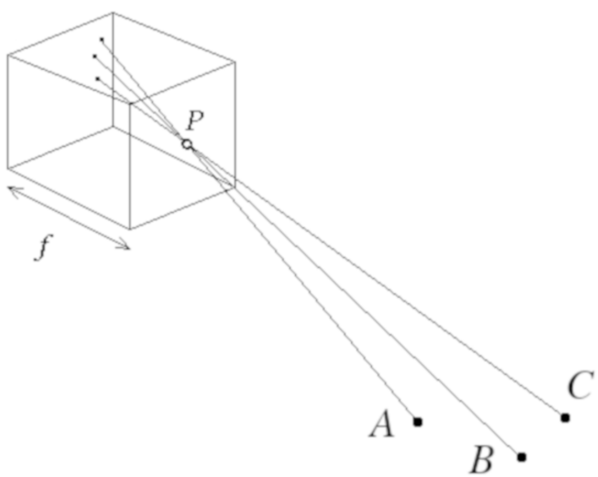
Here is an image showing one of the double toroid, the *DC* and the *CSDC*

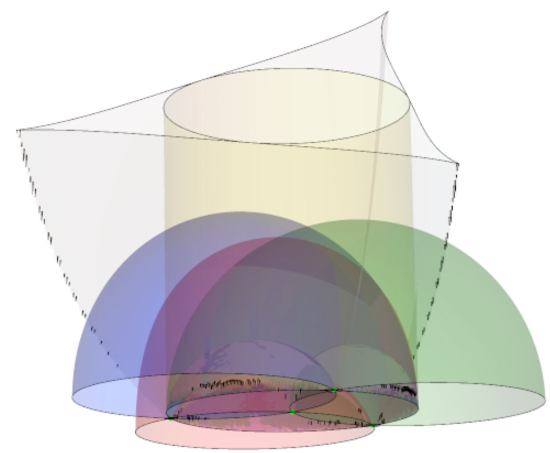
It is barely possible to see the mounds that are part of the *CSDC*

The Danger Cylinder and its Companion Surface

In an effort to answer questions about strongly and weakly related points, it is desirable to understand how the various surfaces involved are interconnected

Some things are pretty easy, like the fact that the danger cylinder is basically inside the *CSDC,* and these intersect along three vertical lines that are equidistant from each other





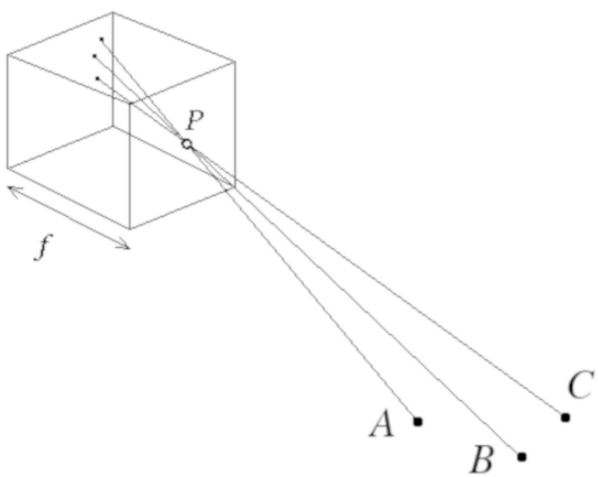
The Danger Cylinder and its Companion Surface

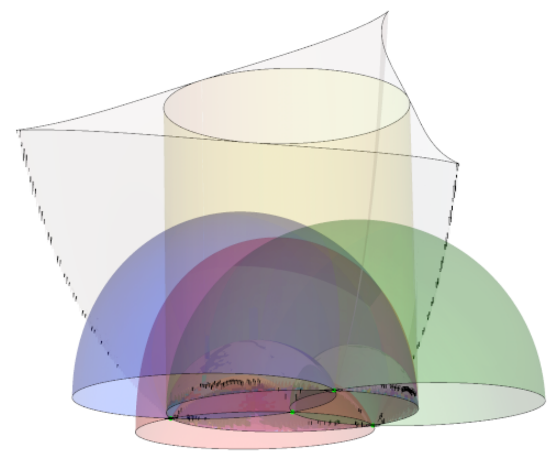
Next, we will consider the intersection of the various double toroids that arise in P3P, namely, *DToroid*$_\alpha$(P), *DToroid*$_\beta$(P) and *DToroid*$_\gamma$(P)

The way that two of our inner toroids (which are similar to American footballs) intersect each other is easily described based on how their tangent cones intersect

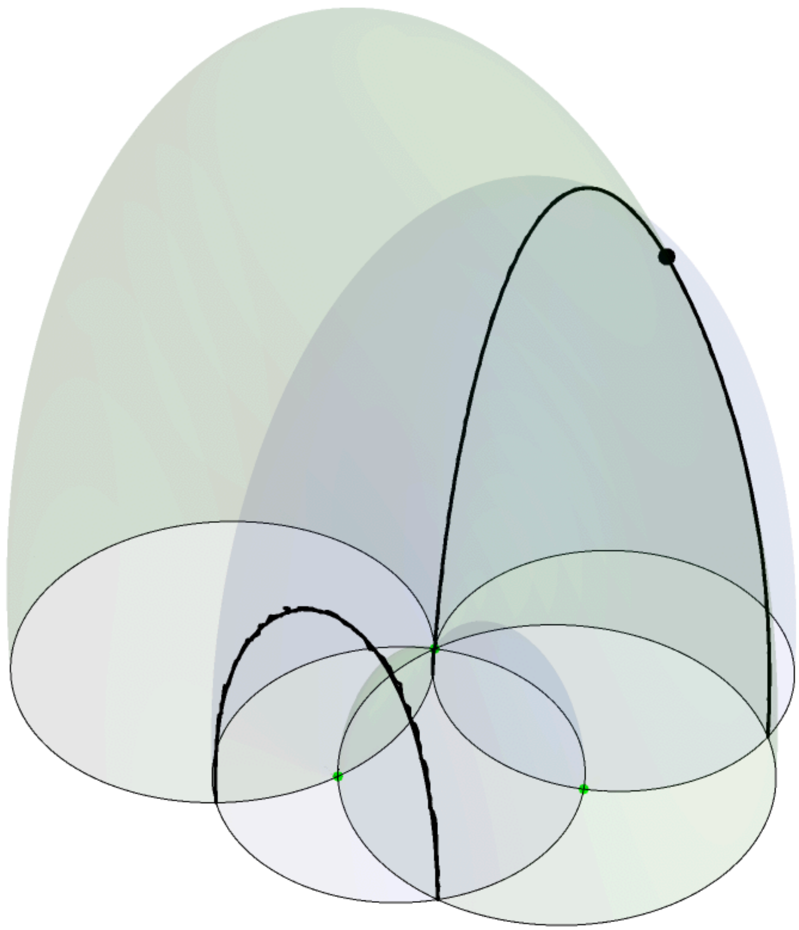
Other toroid intersections are a bit more complicated

Here are three examples of intersecting double toroids:





Intersecting Surfaces

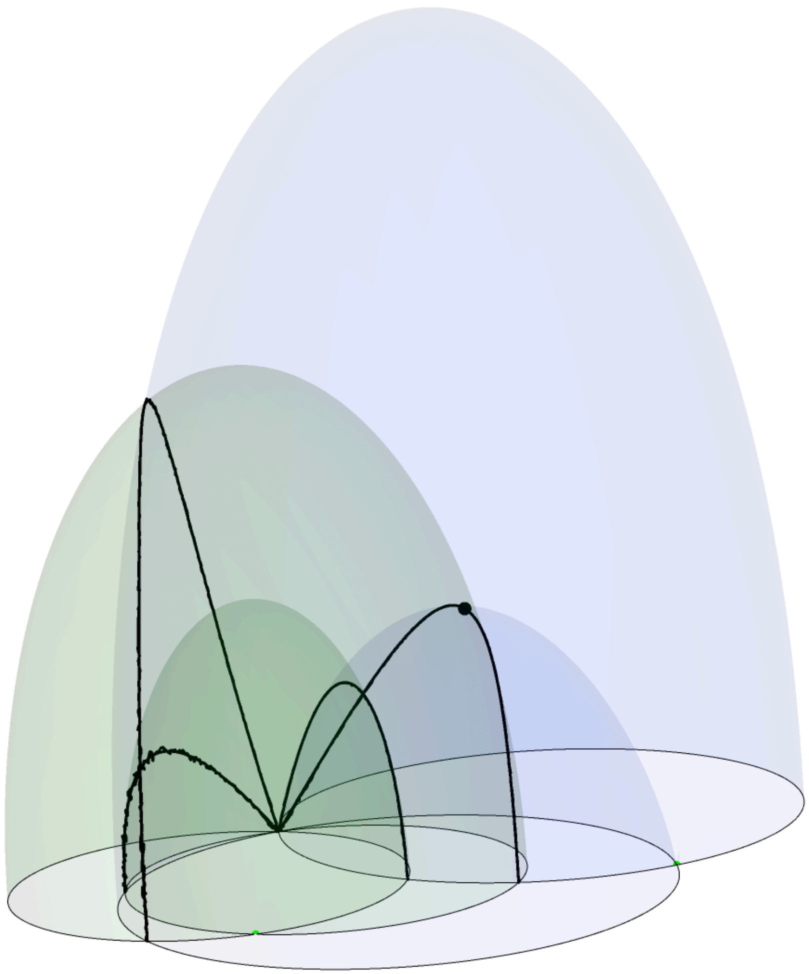


The intersection of two of the double toroids, for some camera's pinhole position

Only the two outer toroids are involved in this example

(Again, only the upper half-space is shown here)

Intersecting Surfaces

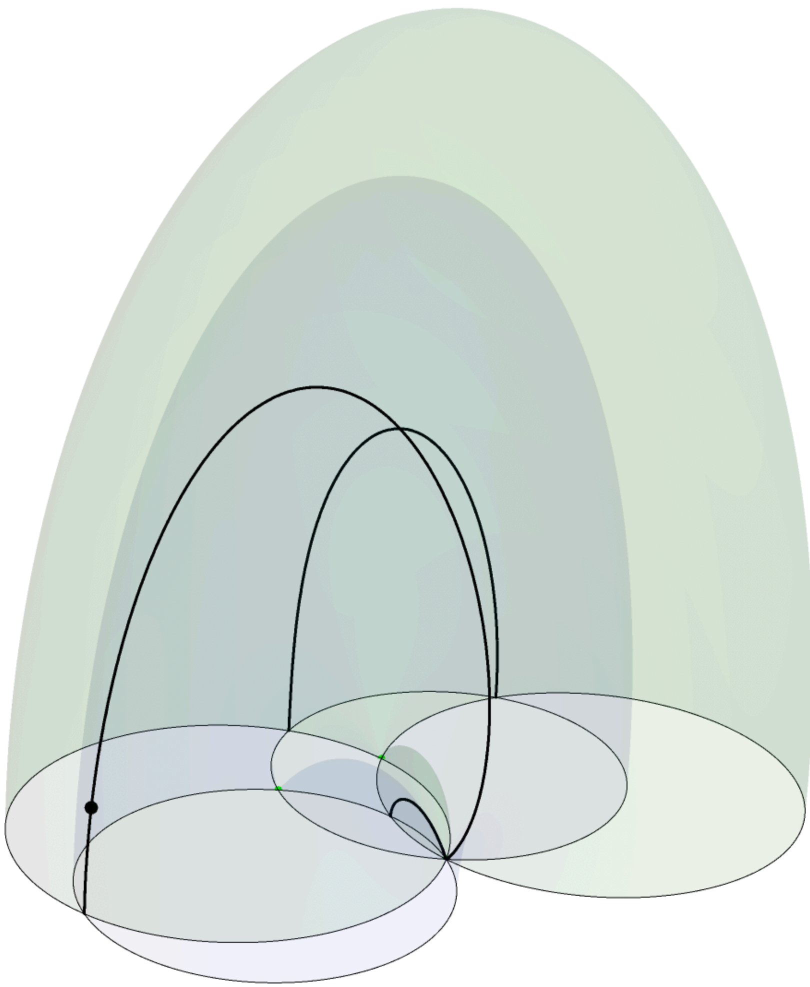


A more complicated intersection of two double toroids

Here the inner toroids get involved

Each of the two toroids that make up the green double toroid intersects each of the two toroids that make up the blue double toroid, in a closed curve

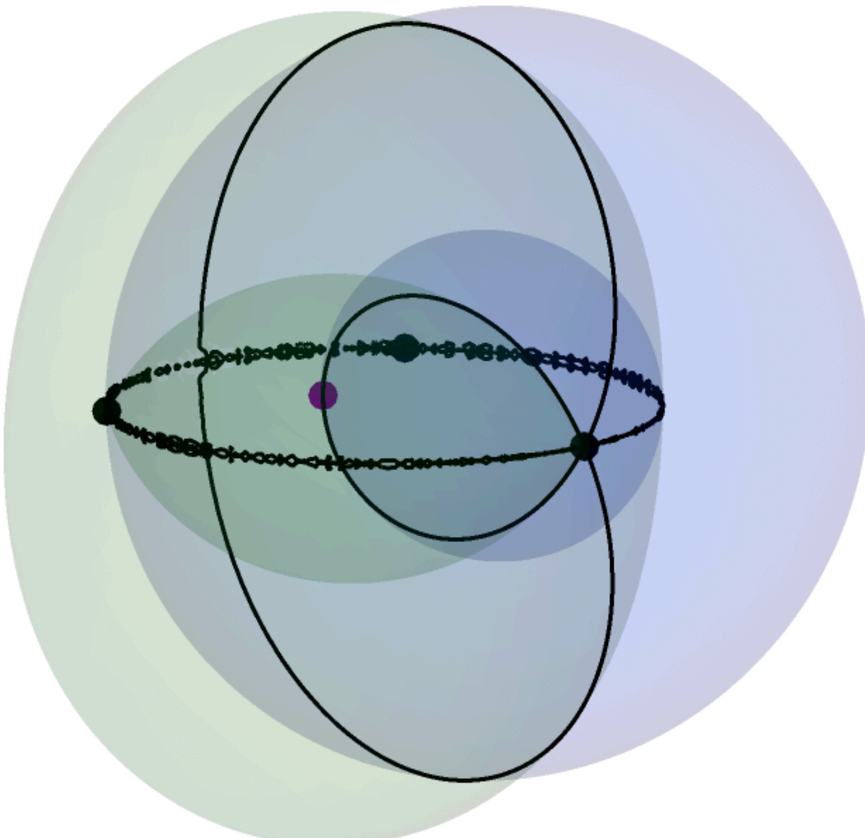
Intersecting Surfaces



Another example of intersecting double toroids

This one is similar to the first example, except that now the two inner toroids intersect in a (small) closed curve

Intersecting Surfaces

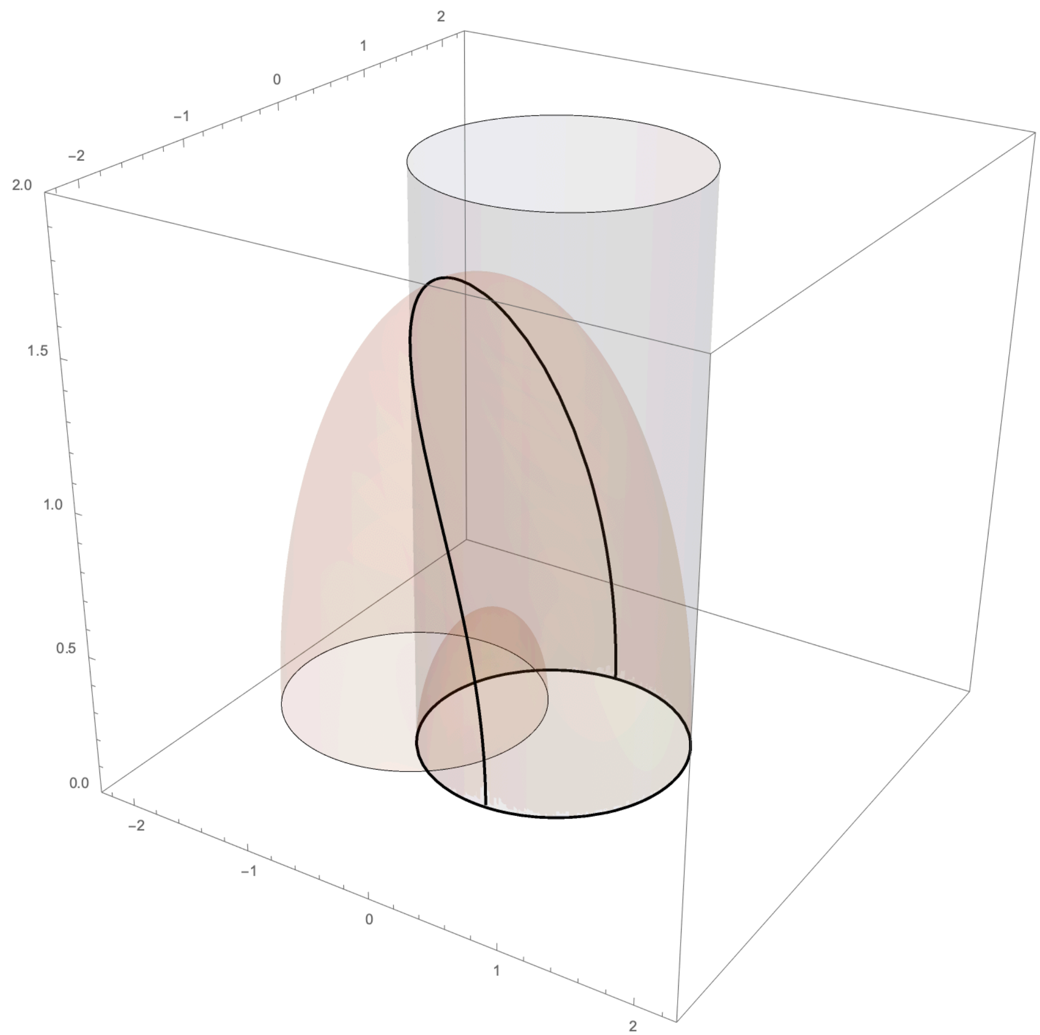


Here we see the intersection of two of the three *special* double toroids, $DToroid_A$, $DToroid_B$, and $DToroid_C$, for an acute triangle

Ignoring the circumcircle (poorly rendered here), the special double toroids seem to always intersect in this simple "double loop" way, at least for acute triangles

All three of the special toroids intersect at the orthocenter (purple dot)

Intersecting Surfaces



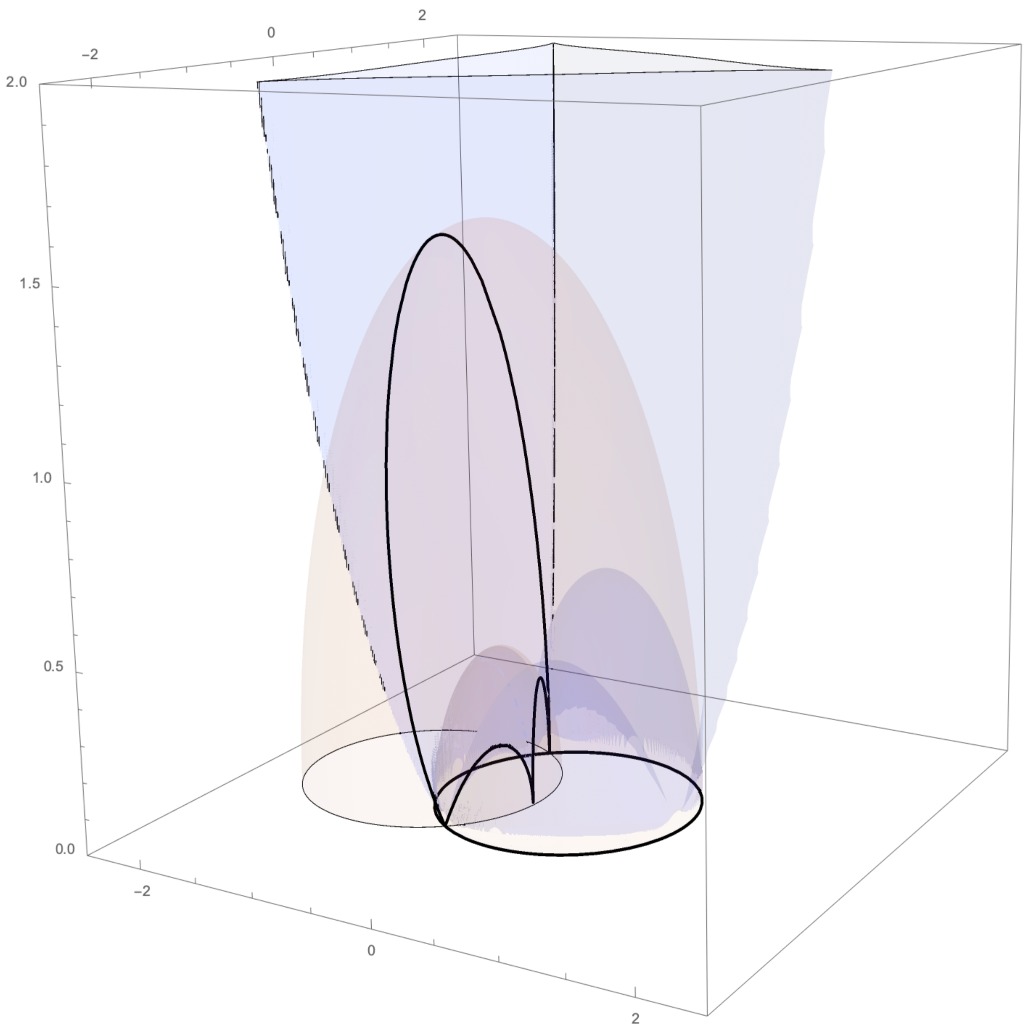
The way in which the danger cylinder intersects one of our special double toroids is rather simple

Here is the danger cylinder together with the double toroid $DToroid_C$ (obtained by rotating the triangle ΔABC's circumcircle about its sideline AB)

The intersection of these two surfaces is the circumcircle and a curve that wraps around the outer toroid, and which is rational as a curve in xyZ-space ($Z = z^2$)

(Only upper half-space shown)

Intersecting Surfaces

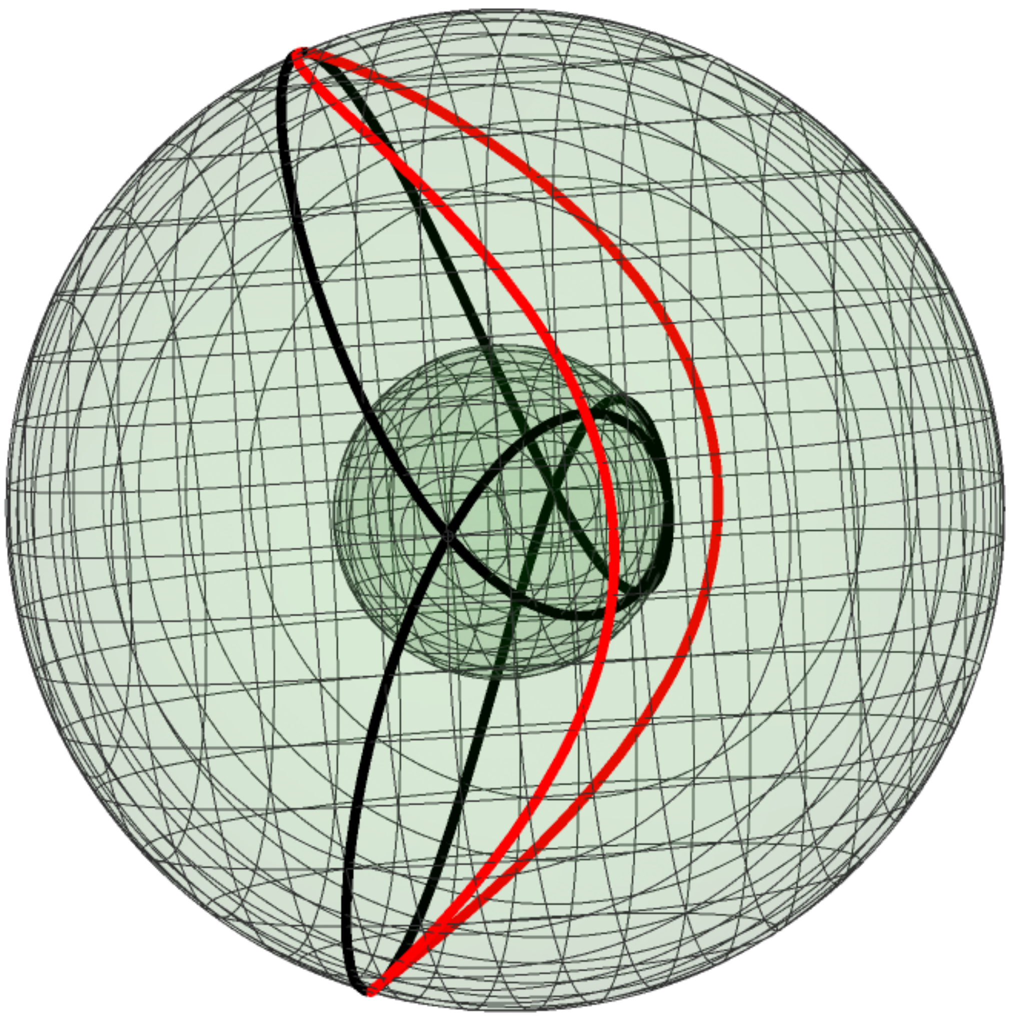


The intersection of $DToroid_C$ and *CSDC* is considerably more complicated than the intersection of $DToroid_C$ and *DC*

However, it too involved the unit circle, plus another close curve that is rational as a curve in xyZ-space

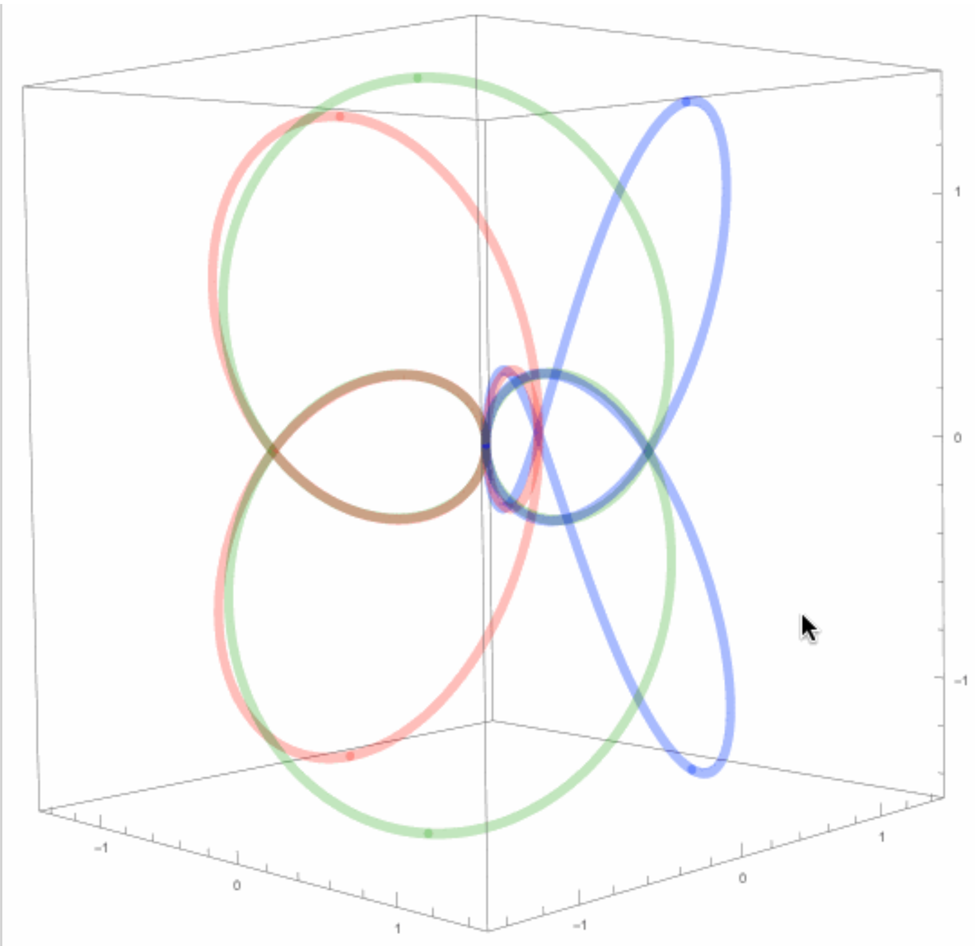
(Only upper half-space shown)

Intersecting Surfaces



For the equilateral triangle case, here is *DToroid*$_{C}$, its intersection with the danger cylinder (red curve), and its intersection with the *CSDC* (black curve)

Intersecting Surfaces



$CSDC \cap (DToroid_A \cup DToroid_B \cup DToroid_C)$ in the equilateral triangle case

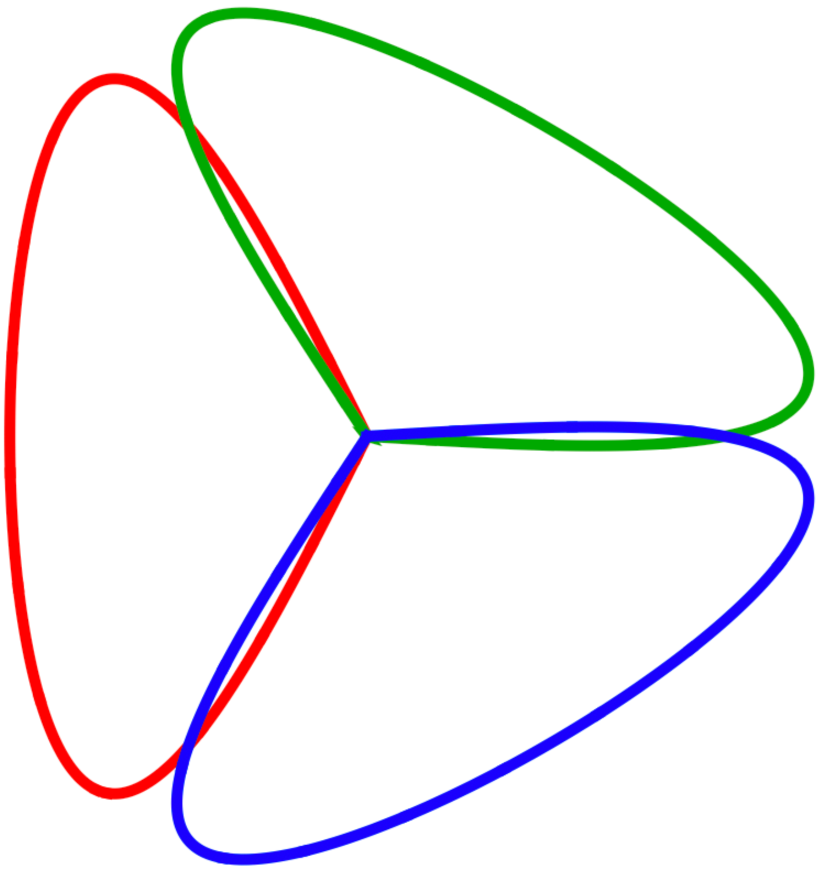
(Wait for animation)

The curves $CSDC \cap DToroid_X$ and $CSDC \cap DToroid_Y$ get very near each other inside the unit sphere, for each pair $\{X, Y\} \subset \{A, B, C\}$

It seems hopelessly messy inside the unit sphere, even for the equilateral triangle case!

This phenomenon will be seen again later in connection with a certain vector field

Intersecting Surfaces



The top-down view of the space curves on the previous slide is remarkably simple, as seen in the image on the left

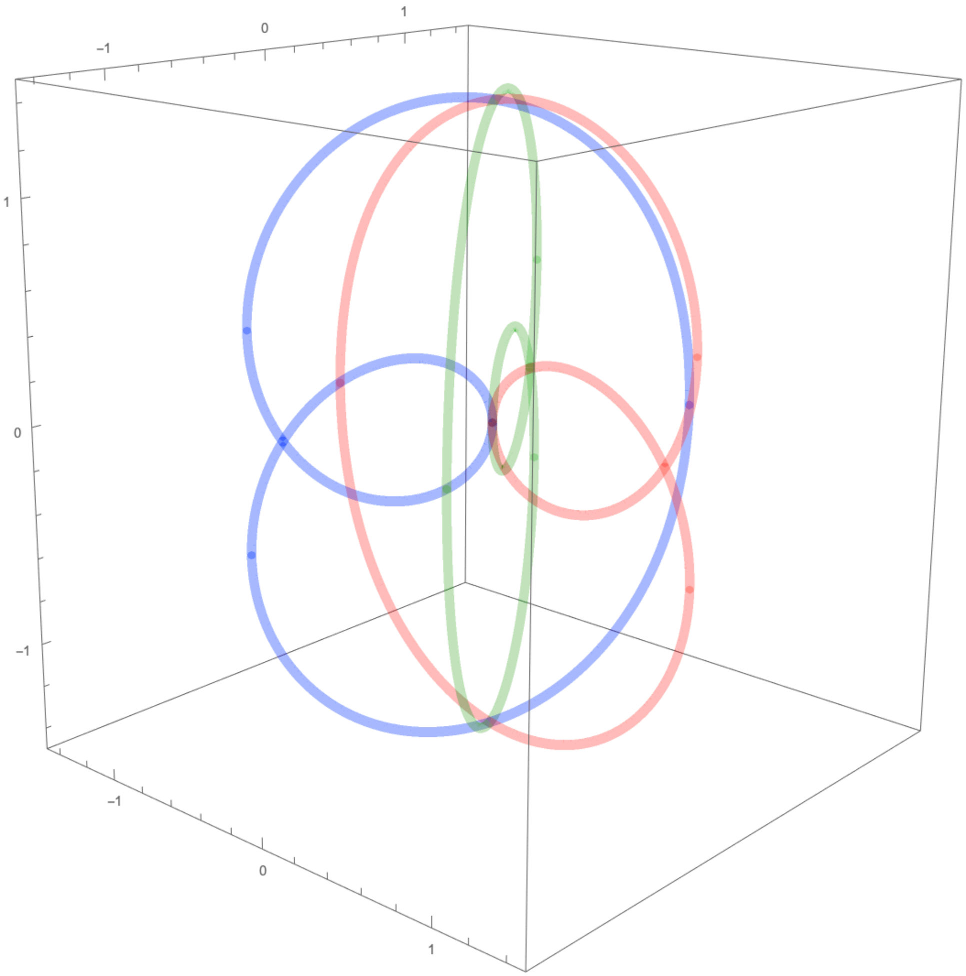
One of the space curves (*CSDC* ∩ *DToroid*$_X$) has this parameterization:

$$\left(x \ , \ \pm 2x\sqrt{(1-x^2)} \ , \ \pm(1+2x)\sqrt{(x^2-x)} \right)$$

where the ±'s are independents

The other two are just rotations of it

Intersecting Surfaces



Pairwise intersections of *DToroid*$_A$, *DToroid*$_B$ and *DToroid*$_C$, in the equilateral triangle case

These intersection are easily understood and visualized, even inside the unit sphere

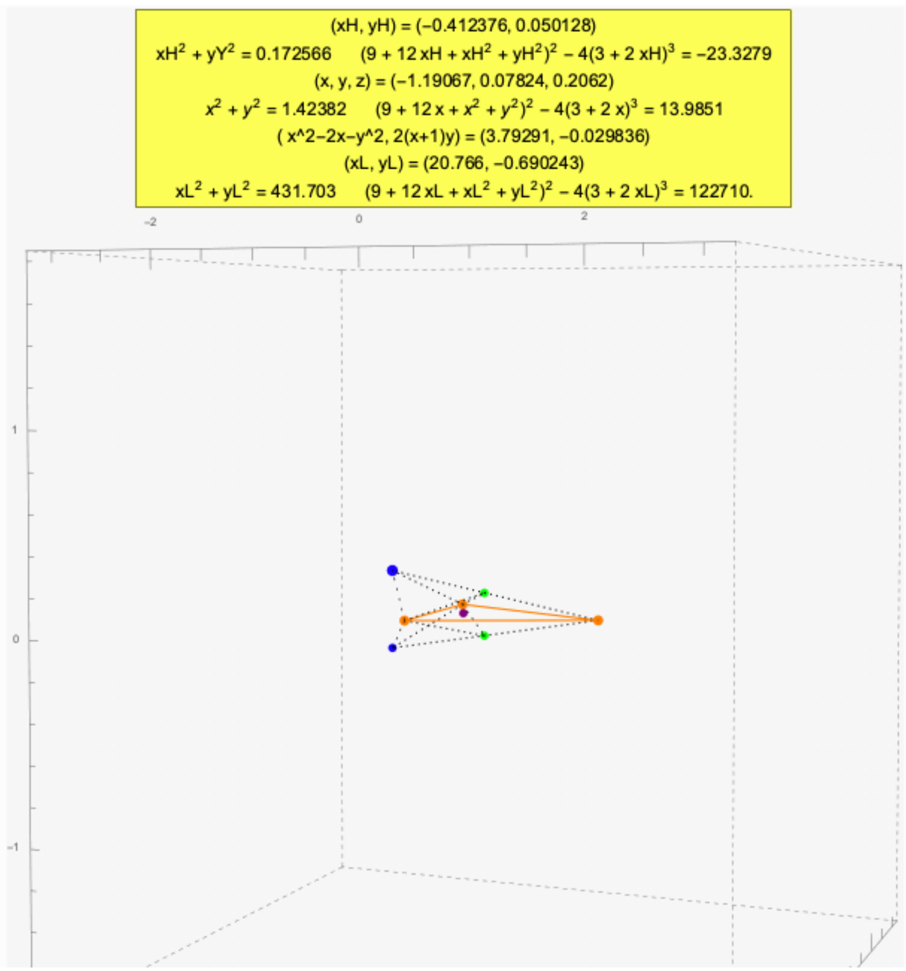
One of these intersection curves has this parameterization:

$$\left(x,\ 0,\ \pm\sqrt{\left[\left(2 + x - 2x^2 \pm \sqrt{(4-3x^2)}\right) / 2 \right]} \right)$$

where the ±'s are independent

The other two are just rotations of it

Intersecting Surfaces



Moving the camera's pinhole around fairly close to the control points

(Wait for animation)

Due largely to all the significant and intersecting surfaces that are fairly close to the control points, when the camera moves around in this region, things get a bit crazy!

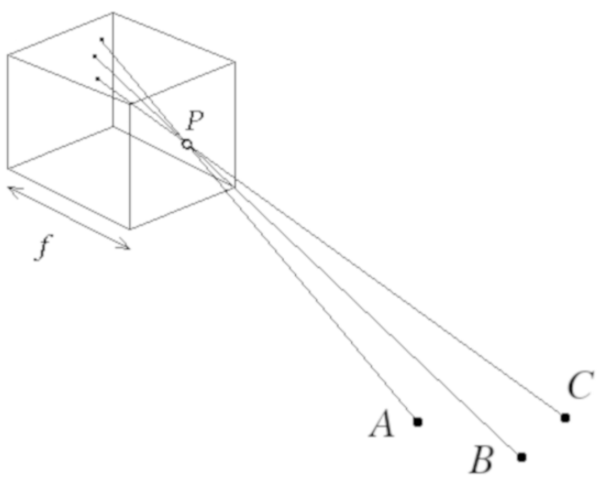
Applying Basic Algebraic Geometry

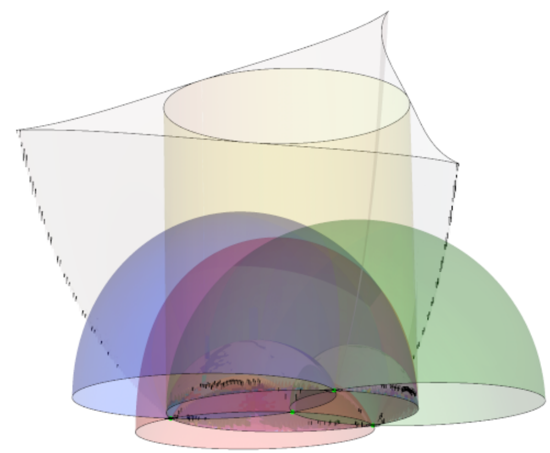
Algebraic geometry is primarily focused on studying systems of polynomial equations in several variables, and the sets of points in space that satisfy such systems

We will here oversimplify this very important area of mathematics

The set of points is called a *variety,* but we may also refer to it as a “space"

A variety has an associated dimension

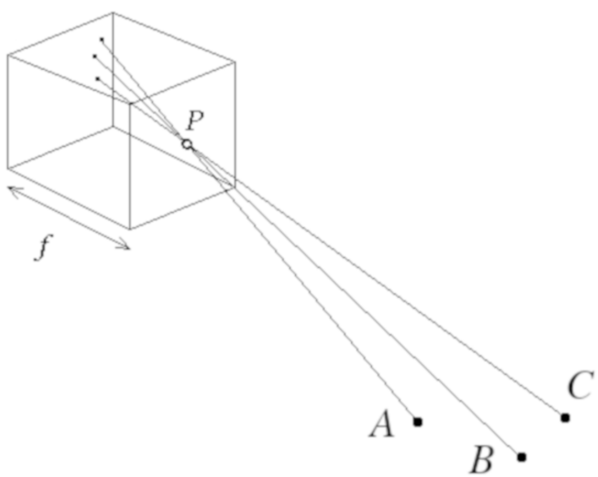


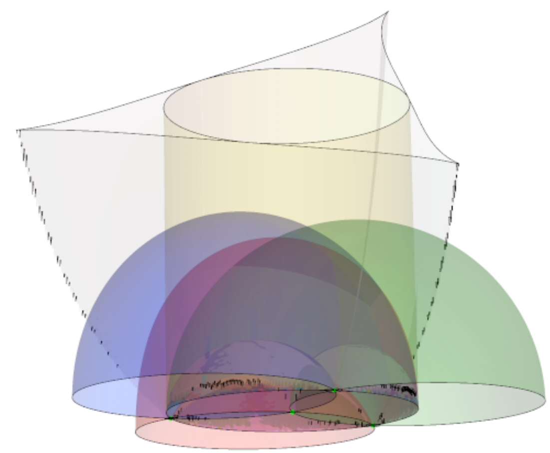


Applying Basic Algebraic Geometry

As an example, for fixed parameters d_1, d_2, d_2, c_1, c_2, c_3, consider Grunert's system of equations in the variables (unknowns) r_1, r_2 and r_3

The solutions are a finite number of isolated points in the 3D-space of all possible triples $(r_1, r_{2,} r_3)$, and so this variety has dimension zero



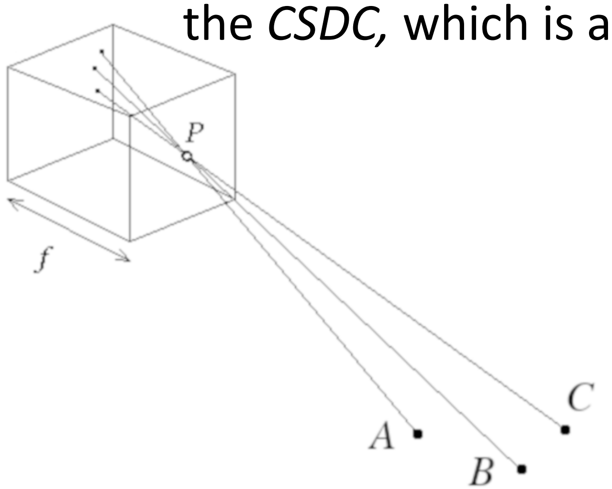


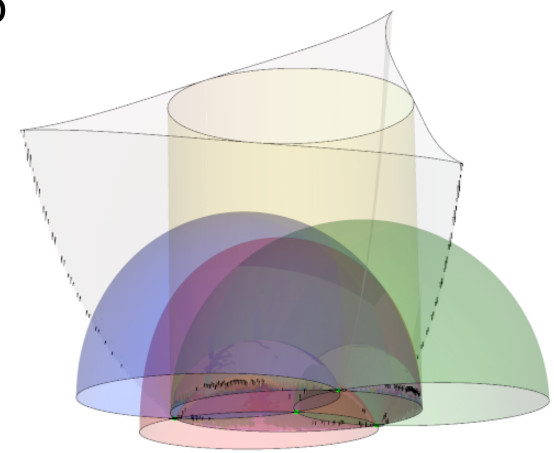
Applying Basic Algebraic Geometry

As another example, consider all of the points in xyz-space for which $(9 + 12\, x_L + x_L^2 + y_L^2)^2 = 4\,(3 + 2\, x_L)^3$, where x_L and y_L are given by certain rational functions of x, y and z (as suggested earlier)

After "clearing out denominators," this yields a single polynomial equation in x, y and z

The variety for this is *CS* ∪ *CSDC, i.e.* the union of the danger cylinder and the *CSDC,* which is a surface, and so its dimension is two





Applying Basic Algebraic Geometry

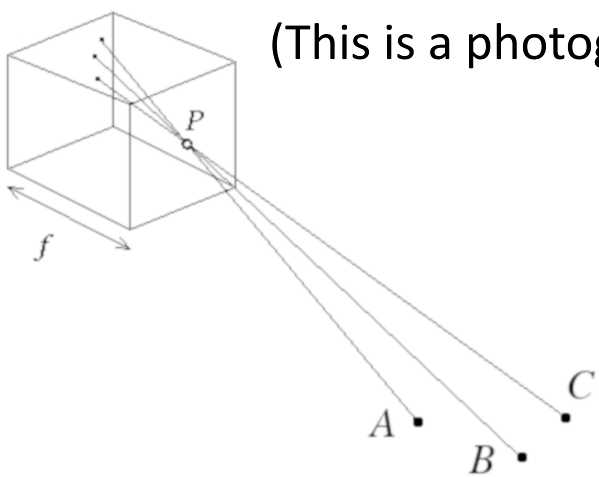
Our double toroids are also examples of varieties of dimension two

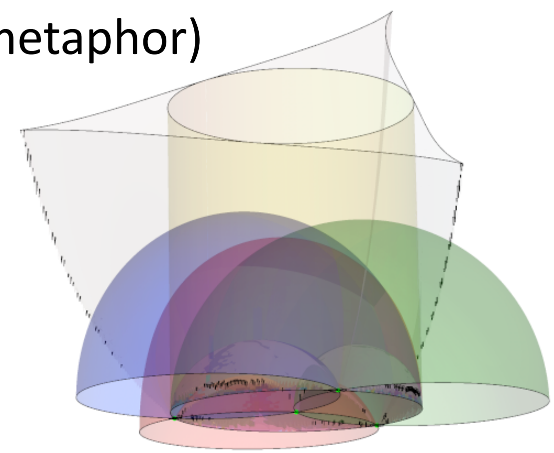
Varieties often have non-smooth points called *singularities*

Each double toroid has two such singularities

Blowing up points is one standard way to tame singularities in algebraic geometry

(This is a photography metaphor, not a demolition metaphor)

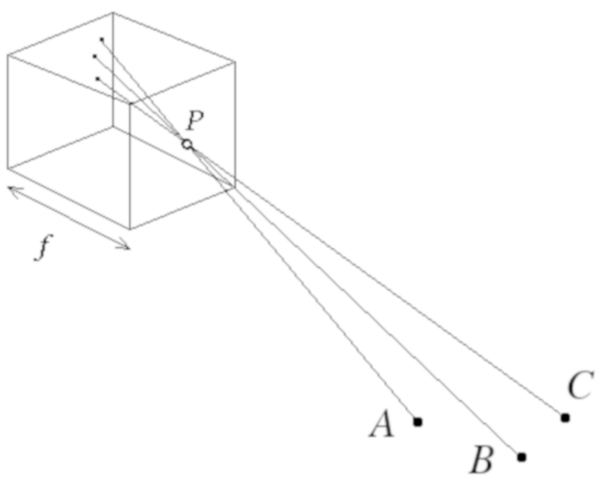


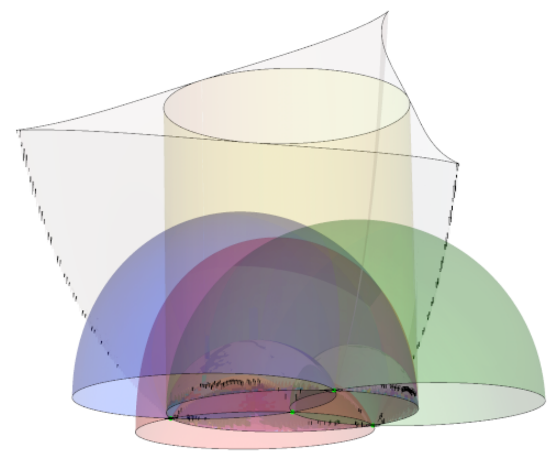


Applying Basic Algebraic Geometry

The idea is that a (singular) point gets "magically" replaced with some higher dimension variety

For instance, the standard way to blowup the point A in xyz-space is to consider the set of all ordered pairs (p, l), where p is a point in xyz-space (3D real space), and l is a line through p and A



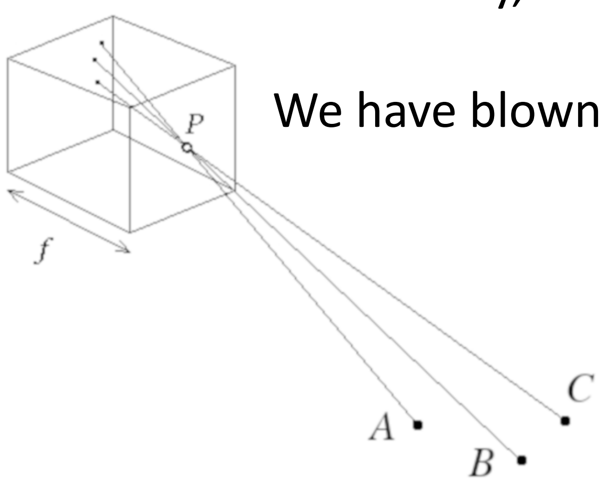


This set of ordered pairs can naturally be given the structure of a variety (space)

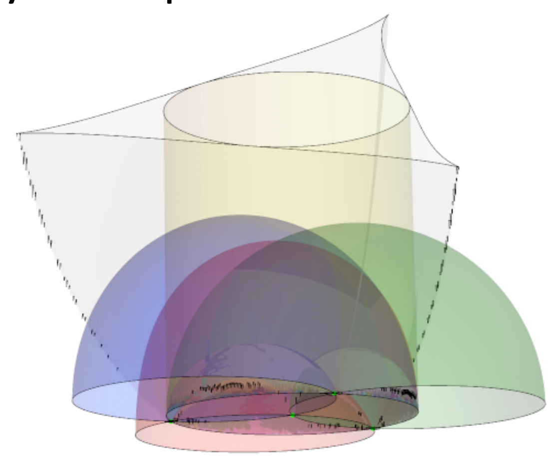
Each point p other than A corresponds to a unique pair (p, l)

But there are infinitely many pairs of the form (A, l) because every line l through A is allowed here

In the new variety, A has been replaced with infinitely many new "points"



We have blown up A!

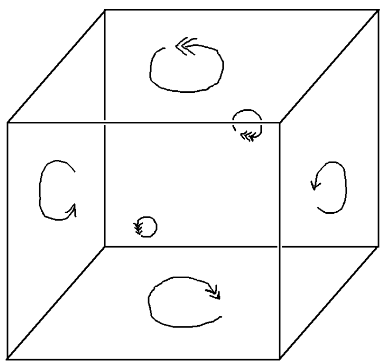


Applying Basic Algebraic Geometry

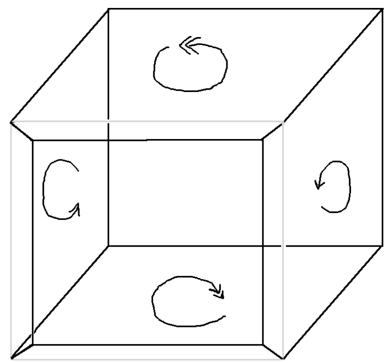
Blowing up A can also be visualized this way:

- From xyz-space, remove a tiny open ball centered at A, leaving a spherical boundary
- Take a copy of real projective 3D space (more about it in a moment) and remove a corresponding open ball from it, leaving a spherical boundary
- Glue the two spherical boundaries together to form a 3D space without a boundary
- This is homeomorphic (topologically equivalent) to the 3D space obtained by blowing up A via the preceding procedure

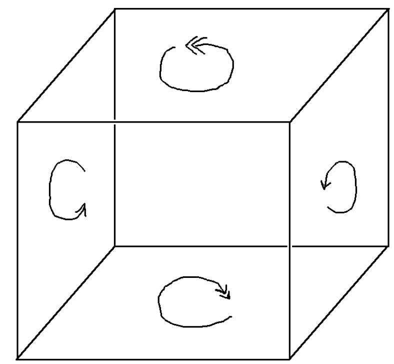
Applying Basic Algebraic Geometry



Projective 3D space can be viewed (topologically) as a solid cube but with points on opposite faces identified in a 180°-rotated manner (this is equivalent to a solid sphere with antipodal points on its spherical boundary identified)



Instead of removing an open ball, we can remove an open solid trapezoidal prism, one that shares the front face of the original solid cube



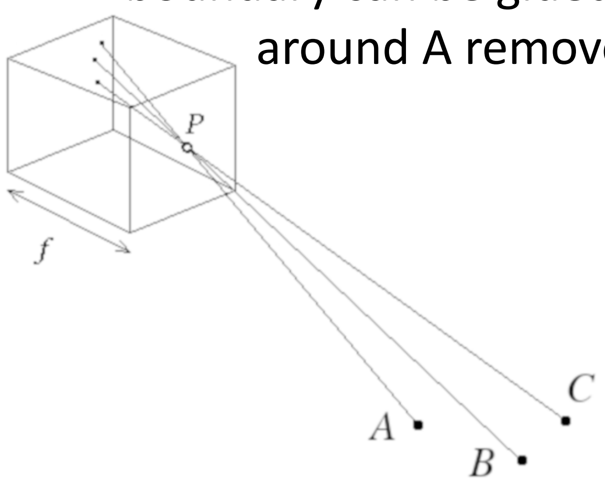
The result can be deformed to a solid cube, with pairs of points identified as before, except for the front and back faces, which are glued together along edges (rotated) to form a surface that is topologically a sphere

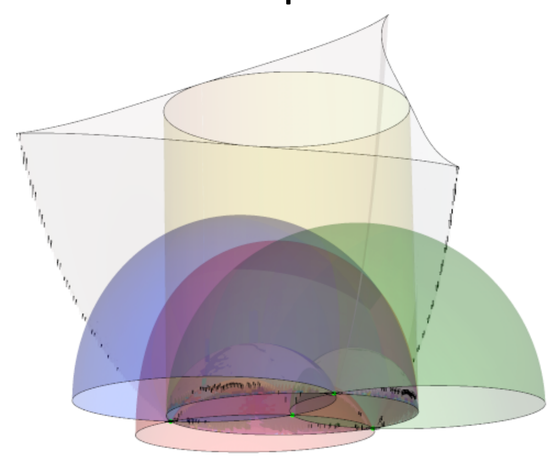
Applying Basic Algebraic Geometry

The space just obtained by essentially removing an open ball from real projective 3D space can be further deformed to produce a finite solid cylinder but with the points on the cylindrical part of its surface identified (in pairs) in a “flipped” manner

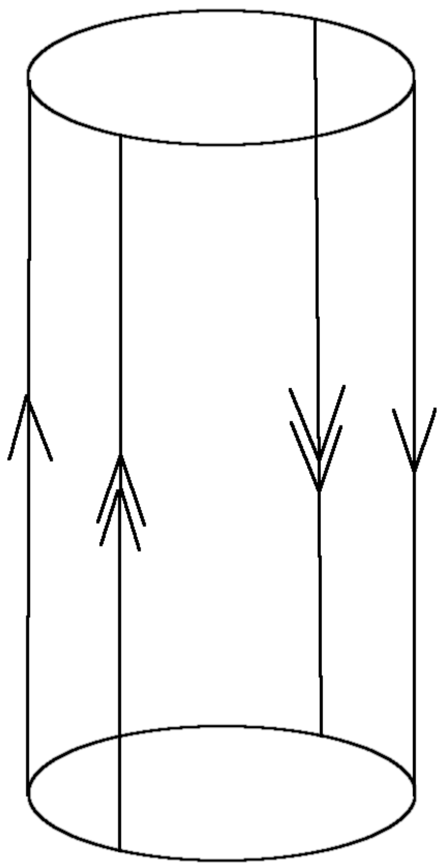
The two disc parts of the surface are glued together along their circular boundaries, but in a rotated manner, to form a sphere, which is the boundary of this space

The above 3D space is somehow a 3D analogue of a mobius strip, and its spherical boundary can be glued to the spherical boundary of xyz-space with an open ball around A removed





Applying Basic Algebraic Geometry



Here is a picture of the model of 3D projective space with an open ball removed, as described in the preceding slide

Each vertical line segment along the cylindrical part of this is identified with the line segment on the opposite side, but flipped

The top and bottom discs thus get glued together, but rotated

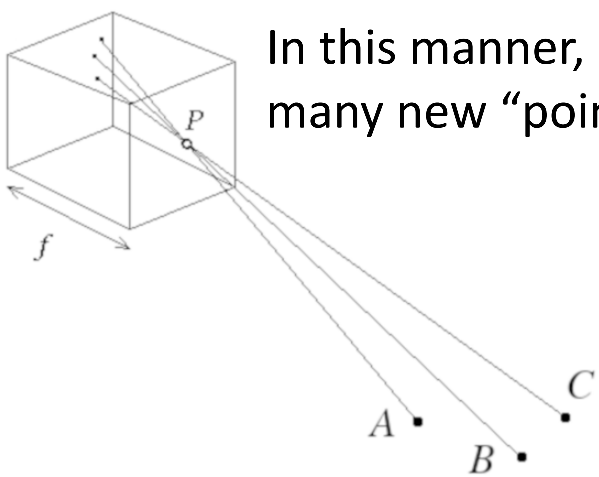
Topologically, they form a sphere

Applying Basic Algebraic Geometry

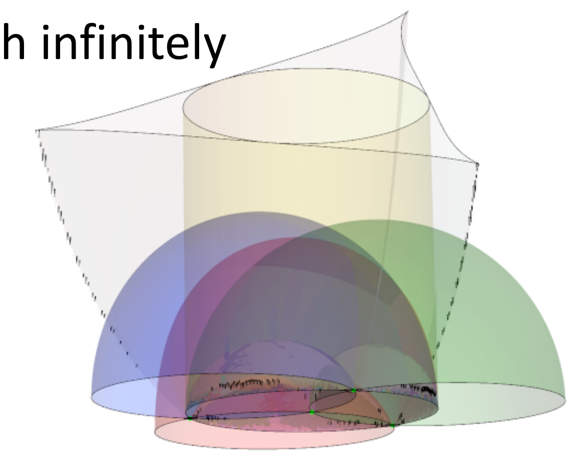
For the purpose of better studying P3P, we will need to blow up all three control points, A, B and C

This amounts to a straightforward embellishment of what we did earlier

Instead of all possible (p, l), use the set of all possible (p, l_A, l_B, l_C), where l_A is a line through p and A, l_B is a line through p and B, and l_C is a line through p and C



In this manner, each of A, B and C gets replaced with infinitely many new "points"

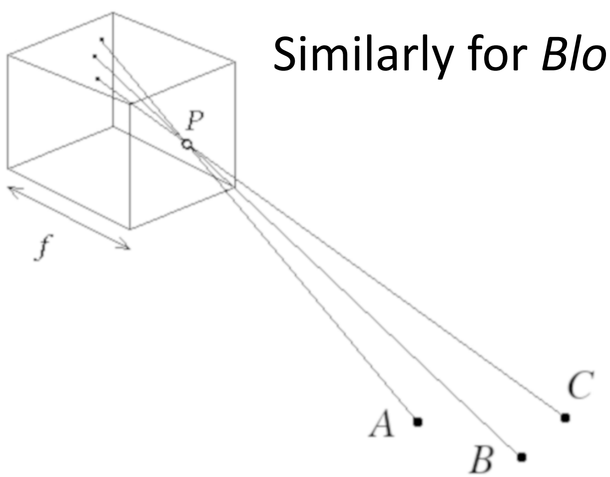


Applying Basic Algebraic Geometry

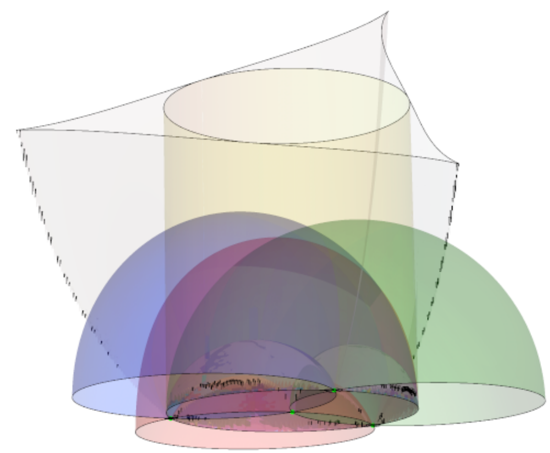
Let *Blowup* denote our new space (variety) of all suitable quadruples (p, l_A, l_B, l_C) , and refer to the quadruples as its "points"

Each point of the form (A, l_A, l_B, l_C) is only distinguished by l_A , which we visualize as specifying a tangent line when we travel through the point A in xyz-space

These points constitute a subvariety of *Blowup* that we will denote $Blowup_A$



Similarly for $Blowup_B$ and $Blowup_C$

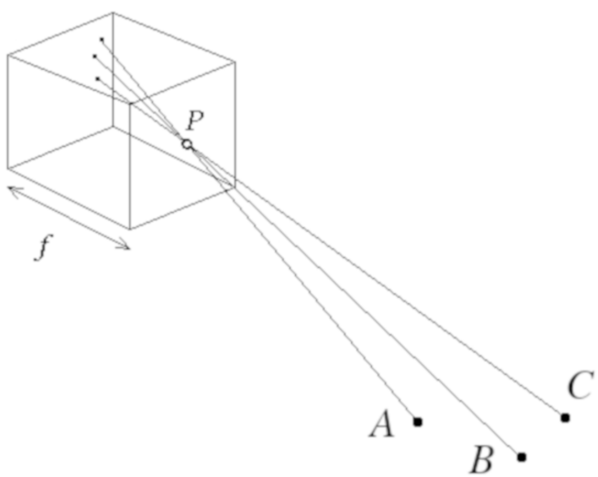


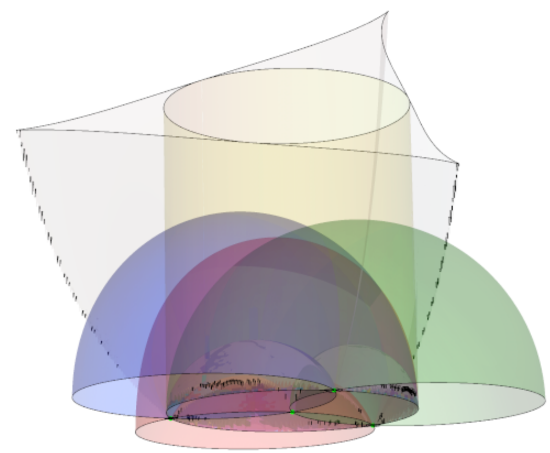
Applying Basic Algebraic Geometry

Next, for any variety V in xyz-space, such as one of our double toroids, we can obtain a corresponding variety in *Blowup*, as follow:

1. Each point of V, other than A, B and C (if part of V), corresponds to a unique point in *Blowup*
2. Take all of these points in *Blowup*, and extend this set by including any of its limit points (topological closure)

This will be a subvariety of *Blowup*, and any limit points that might get "tossed in" will come from the union of $Blowup_A$, $Blowup_B$ and $Blowup_C$



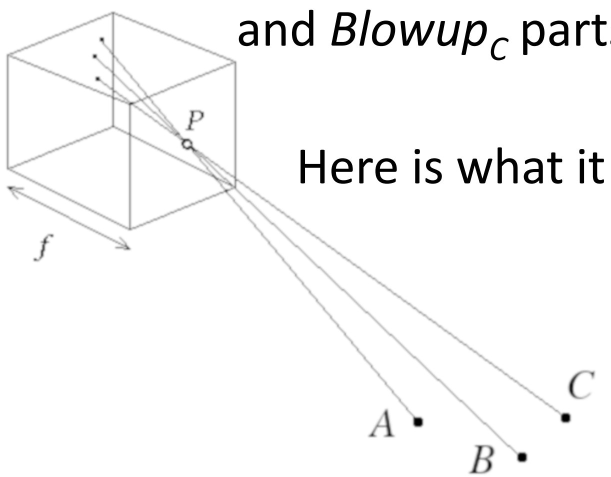


Applying Basic Algebraic Geometry

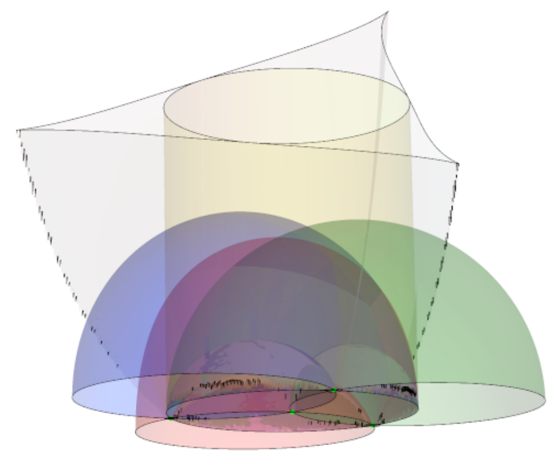
When this technique is applied to one of our double toroids, the resulting subvariety of *Blowup* is a ***torus***, which of course is non-singular everywhere

As seen in earlier animations, the surface *CSDC* has one-dimensional singularites (like the creases along its sides), and these are not “resolved” by simply switching to the corresponding surface in *Blowup*

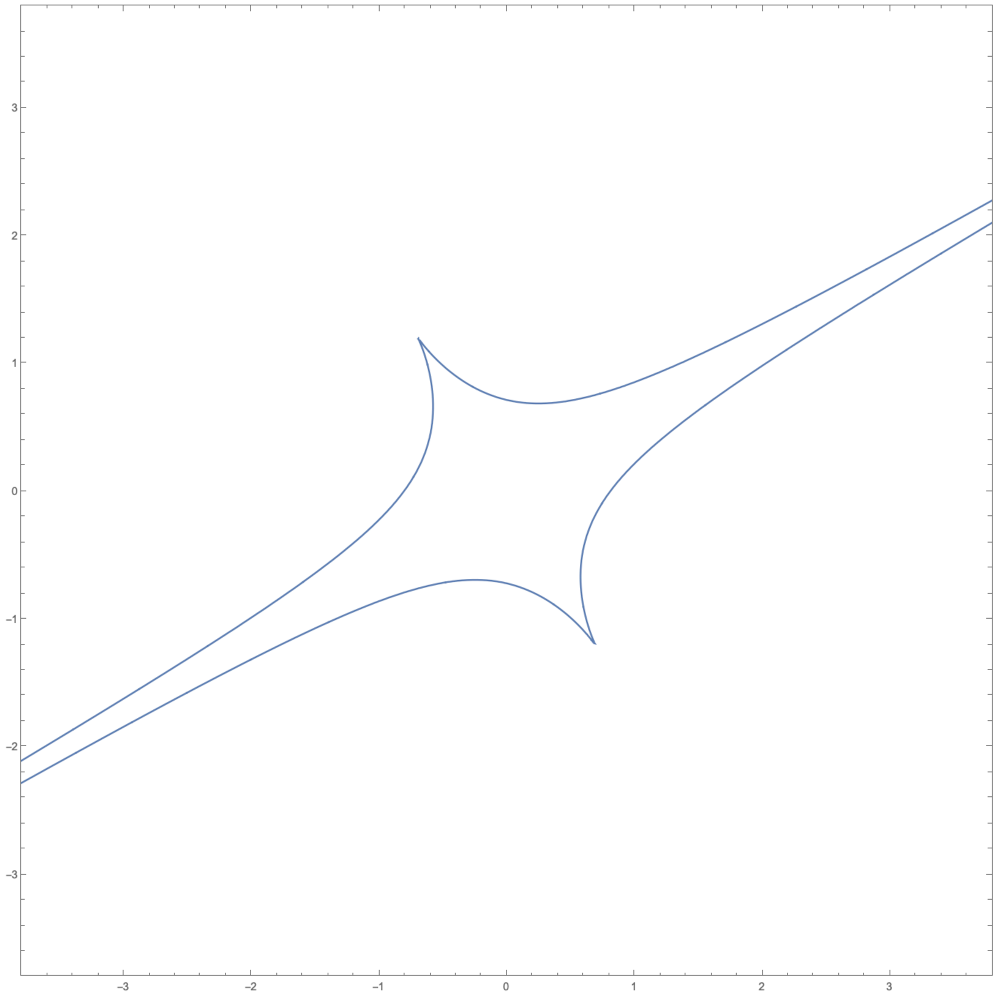
Yet, it is useful to look at this surface in *Blowup*, and in particular, the $Blowup_A$, $Blowup_B$ and $Blowup_C$ parts



Here is what it looks like at one of these:



Applying Basic Algebraic Geometry



The intersection of *Blowup*$_C$ and *"CSDC"* in *Blowup*

Blowup$_C$ can be thought of as a space whose points correspond to the lines (in 3D space) through C

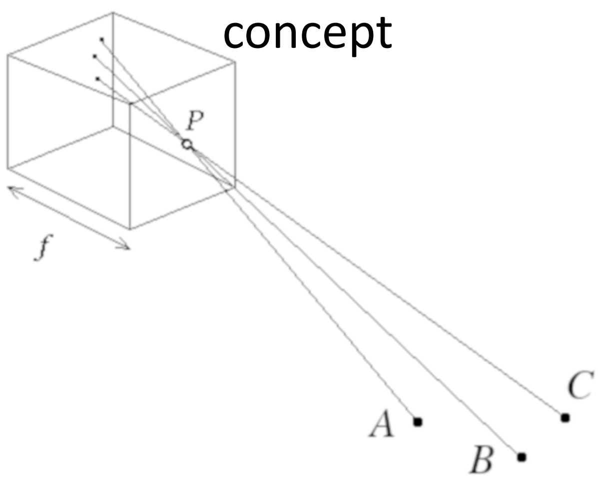
Blowup$_C$ is therefore a *real projective plane,* a variety that can also be visualized as an ordinary (affine) plane with a *projective line* of extra points added "at infinity"

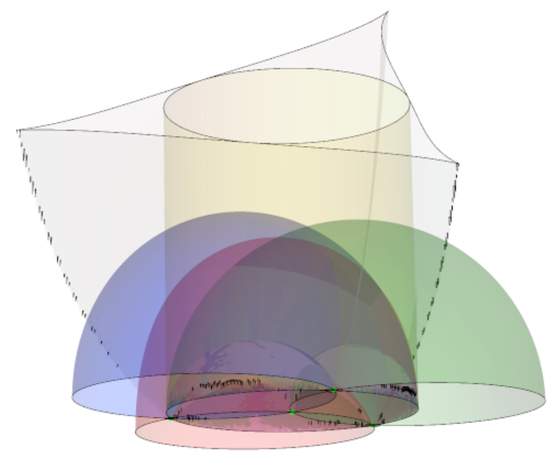
Applying Basic Algebraic Geometry

Here is one reason for using the variety *Blowup* in connection with P3P: The quantities c_1^2, c_2^2, c_3^2 and $c_1c_2c_3$ all extend naturally to all of the points of *Blowup*

Moreover, there is a map from *Blowup* onto the variety *V* of all points (q, r, s, t) for which $q^2 = rst$ that sets $q = c_1c_2c_3$, $r = c_1^2$, $s = c_2^2$ and $t = c_3^2$

Points of *Blowup* are weakly related if and only if they have the same image under this map from *Blowup* to *V*, so we may regard "weakly related" as an algebraic geometric concept





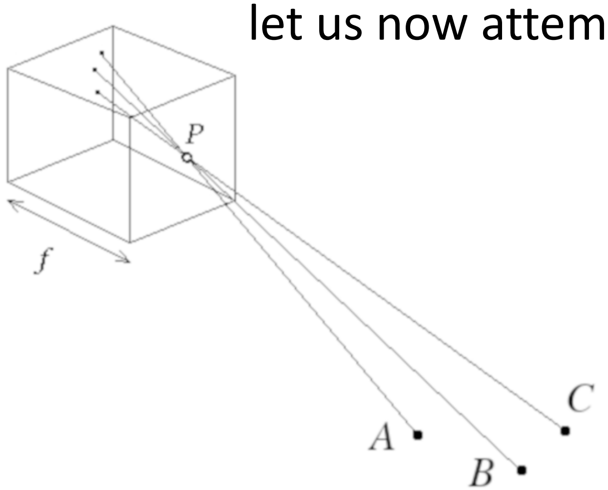
Applying Basic Algebraic Geometry

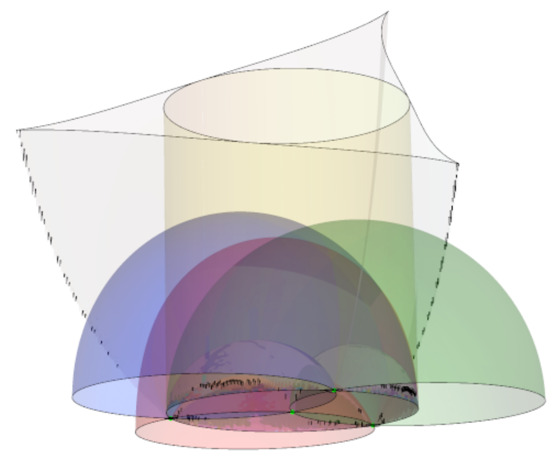
Maienschein and Rieck, working on the 2D version of the P3P Problem, and related issues, discovered that by blowing up A, B and C, in the plane, and then contracting the circumcircle to a point, while also adding one point at infinity, one obtains a (real) ***torus***

This was realized via a birational map, and proves a theorem of W. von Dyck

For this, Maienschein analyzed the situation in terms of directed angles, modulo π

It has been an ongoing frustration that this work has not easily extended to 3D P3P, but let us now attempt such an extension anyway





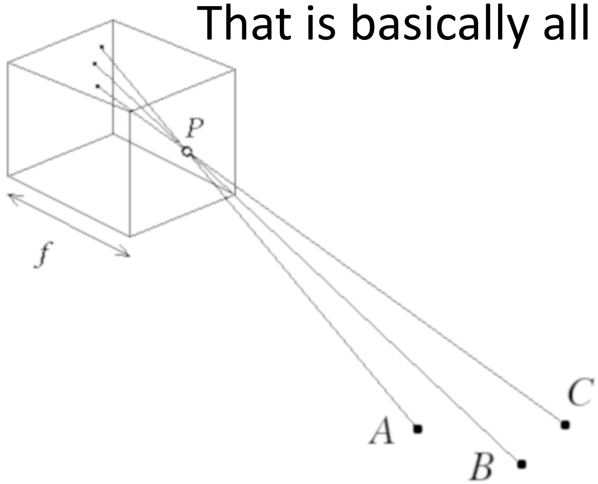
Applying Basic Algebraic Geometry

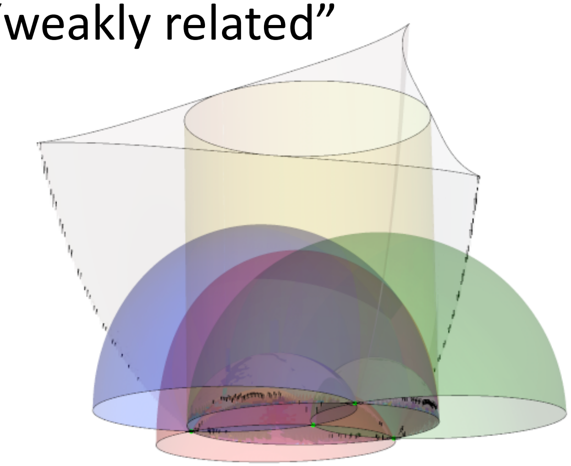
Back to the 3D P3P Problem now

Because c_1^2, c_2^2 , c_3^2 and $c_1c_2c_3$ are constant on the circumcircle, one is inclined to wonder what happens to *Blowup* if the circumcircle is again contracted to a point

Perhaps the resulting space might turn out to have a nicer description that can be more easily visualized

That is basically all there is to say for now concerning "weakly related"





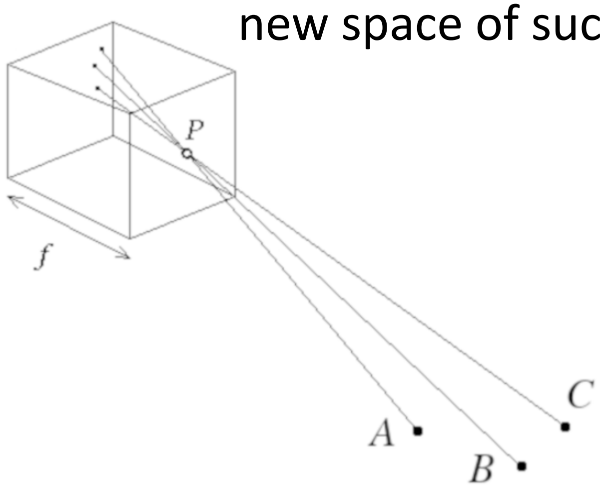
Applying Basic Algebraic Geometry

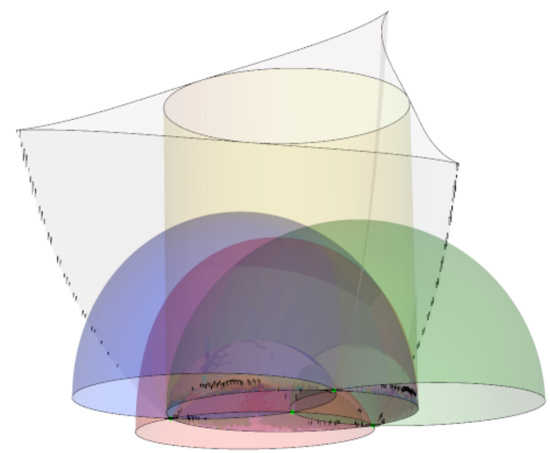
What about "strongly related?"

Unfortunately, c_1 and c_2 are not defined on $Blowup_C$, and so forth, and this is basically why on *Blowup,* we have continuity of "weakly related," but not "strongly related"

One way to "correct" for this is a bit messy, and amounts to replacing the lines l_A , l_B and l_C in (P, l_A, l_B, l_C) with ***directed*** lines, defining α, β and γ to be the angles between pairs of these directed lines, and letting c_1, c_2 and c_3 be the corresponding cosines

This means each ordinary point P in xyz-space gets replaced by ***eight*** points in the new space of such quadruples (P, l_A, l_B, l_C) -- *Ouch!*



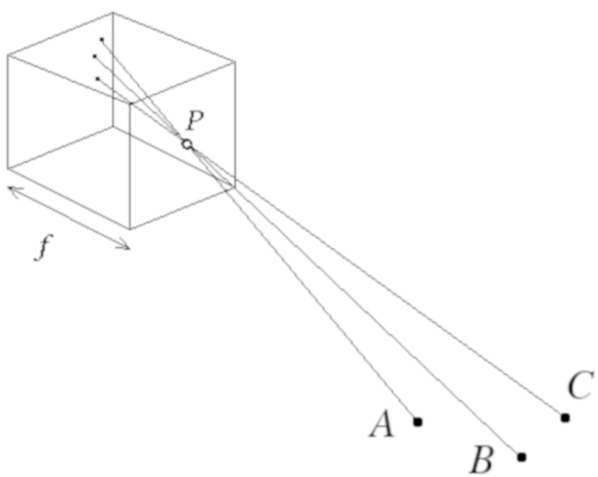


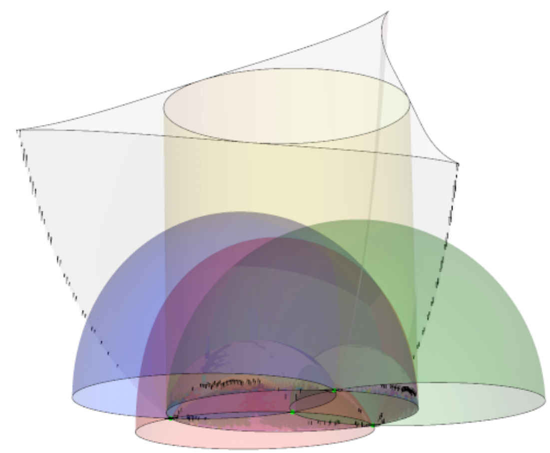
Applying Basic Algebraic Geometry

We say that the variety *Blowup** thus constructed is an 8-fold cover of *Blowup*

We can imagine obtaining this space from eight copies of xyz-space, by extracting tiny open balls around A, B and C, from each copy, and gluing these together appropriately along the spherical boundaries of the balls

The 8-fold cover can easily be reduced to a 4-fold cover by identifying each point (P, l_A, l_B, l_C) with the point $(P, -l_A, -l_B, -l_C)$, where l_A and $-l_A$ are oppositely oriented directed lines based on the same undirected line, and so forth





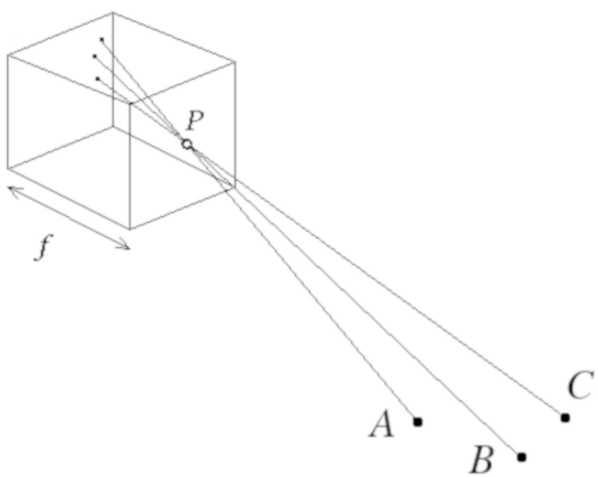
Applying Basic Algebraic Geometry

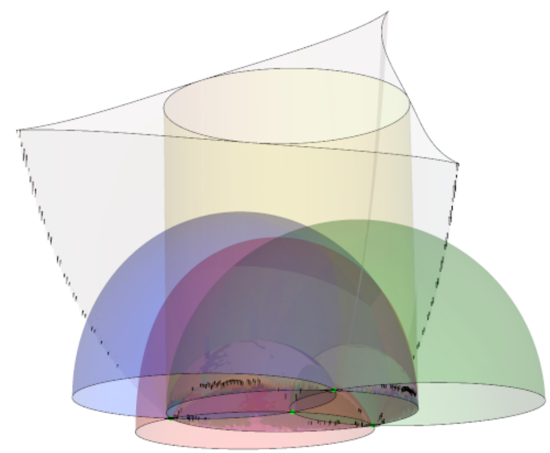
Denoting this 4-fold cover by *Blowup*#, we have a variety on which the quantities c_1, c_2 and c_3 can be naturally defined (the same may be said of *Blowup**)

These quantities vary continuously as a point moves continuously around in *Blowup*#

There is a natural projection of the points in *Blowup*# to the points in *Blowup*

However, a continuous open path in *Blowup*# may project onto a closed path in *Blowup*, thus showing that *Blowup*# is a non-trivial cover of *Blowup*, in the sense of homotopy





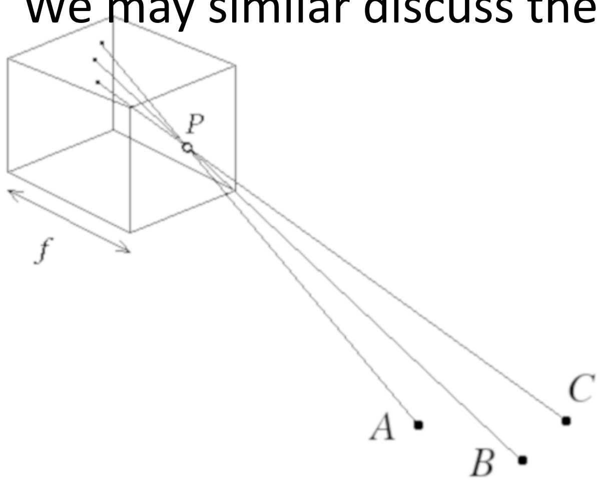
Applying Basic Algebraic Geometry

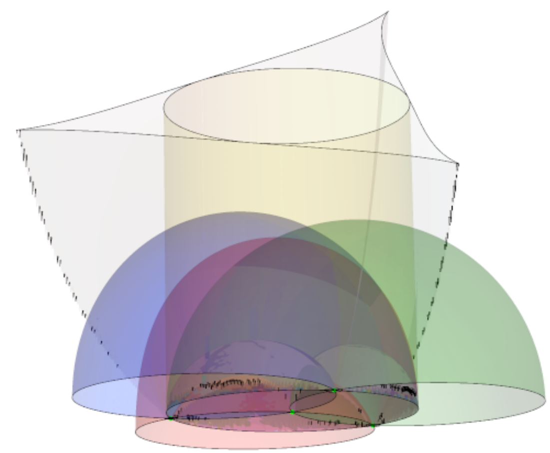
By analogy with classical planar geometry, we regard that each point of *Blowup** has three *angular coordinates* (α, β, γ), where again, α is the angle between the directed lines l_B and l_C, and so forth ($0 \le \alpha \le \pi$, $0 \le \beta \le \pi$ and $0 \le \gamma \le \pi$)

We continue to take $c_1 = \cos\alpha$, $c_2 = \cos\beta$ and $c_3 = \cos\gamma$

Near a general point, (α, β, γ) and (c_1, c_2, c_3) serve as local coordinate systems

We may similar discuss these things for the points of *Blowup*#



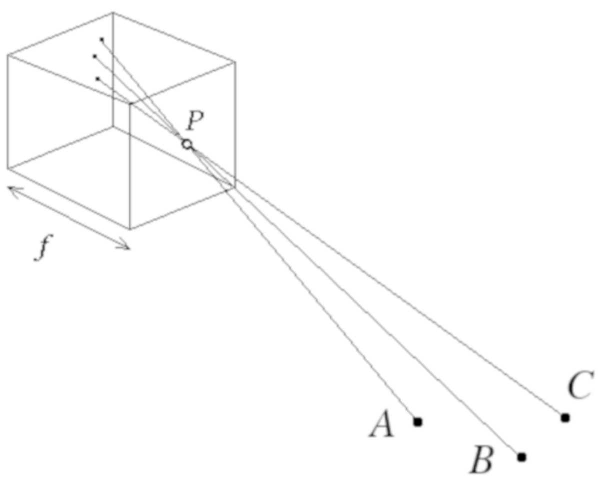


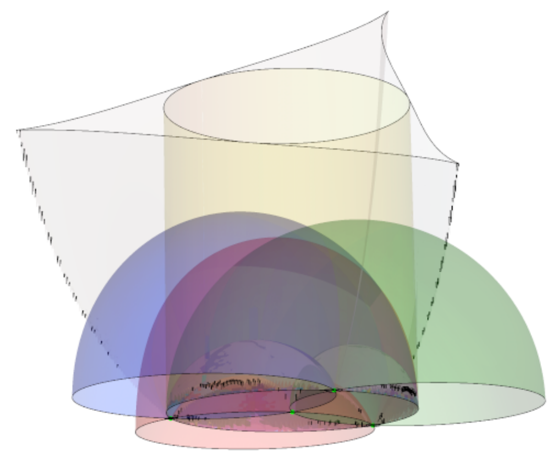
Applying Basic Algebraic Geometry

It can be shown that *Blowup*# contains four copies of the circumcircle on which the angular coordinates are constant

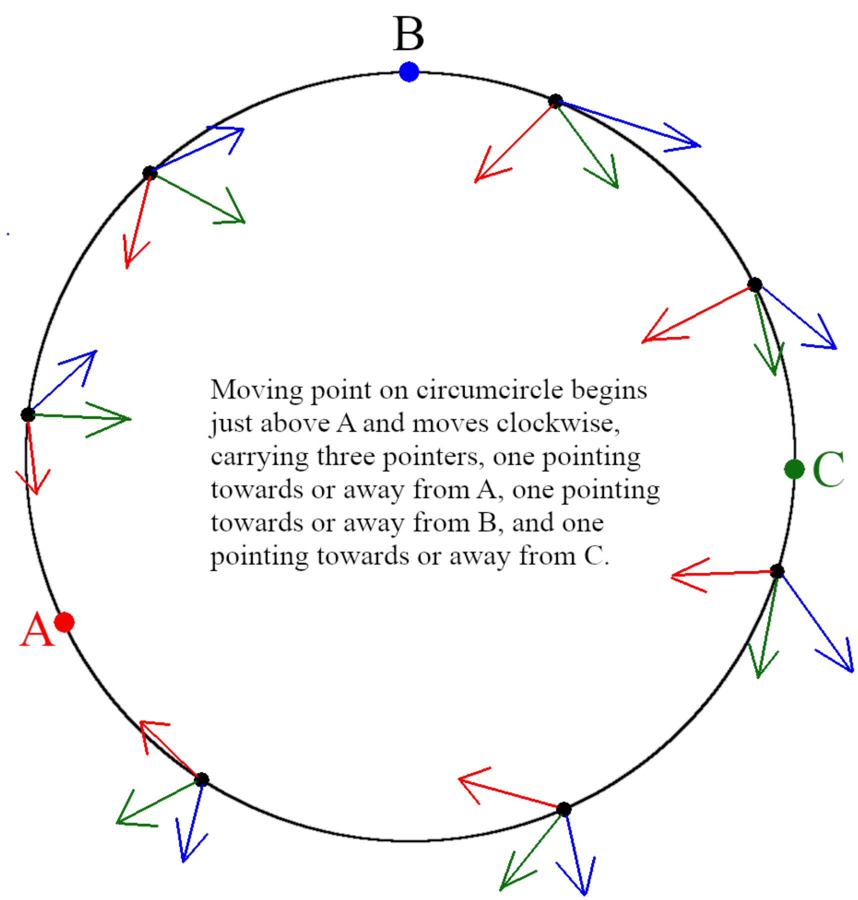
By analogy with the 2D work, it would seem to make sense to collapse each of these circles to a single point (via a "blowdown," the opposite of a blowup)

This yields four special points, and we will denote the resulting space by *Blowup†*, and observe that (α, β, γ) and (c_1, c_2, c_3) are naturally defined here as well





Applying Basic Algebraic Geometry



Representing a lift of the circumcircle to *Blowup or *Blowup*#**

When working in *Blowup**, a lift of a moving point on the circumcircle must go twice around the circle to return to its starting position

When working in *Blowup*# instead, only need one trip around the circle

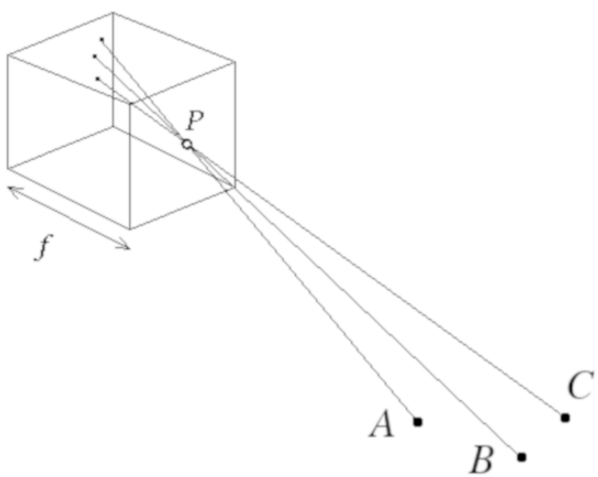
The circumcircle thus lifts to a circle in *Blowup*#, in fact ***four*** such circles, which can be collapsed to make *Blowup†*

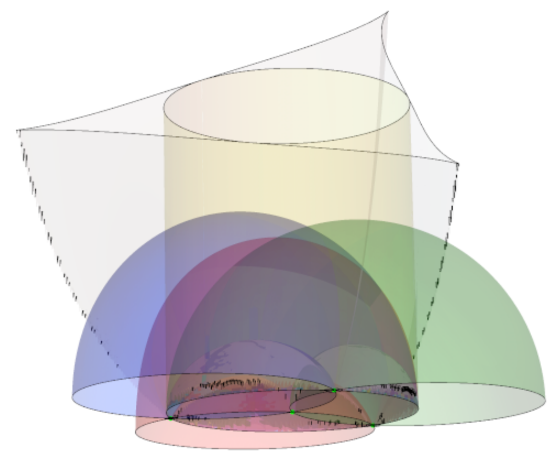
Applying Basic Algebraic Geometry

Extending our terminology, two points of *Blowup†* are "strongly related" if they have the same values for (α, β, γ), or equivalently, the same values for (c_1, c_2, c_3)

Similarly we can define "weakly related"

The four new special points are only weakly related to each other, but each of these points is strongly related to one of the four lifts of the orthocenter H to *Blowup†*

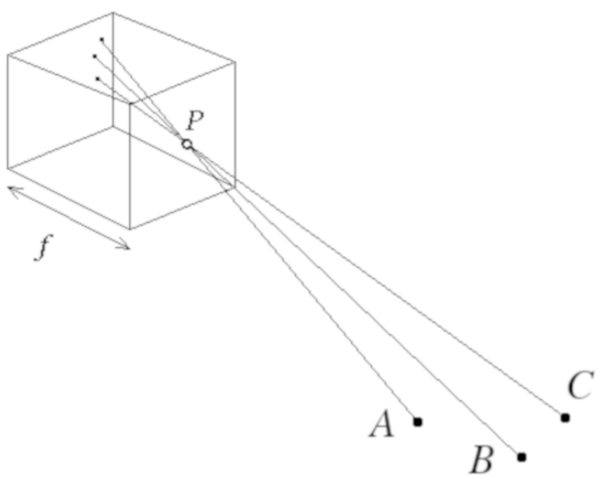


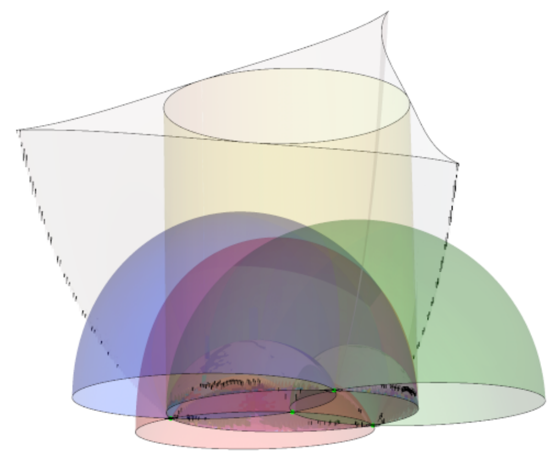


Applying Basic Algebraic Geometry

Before attempting to visualize the 3D-space *Blowup*†, it might make sense to first try to visualize its intersection with the xy-plane, calling this *Blowup*$_{xy}^{\dagger}$

We can imagine beginning with four copies of the xy-plane, removing tiny open discs (of the same size) from around A, B and C, and gluing the circular boundaries of corresponding discs appropriately to obtain *Blowup*$_{xy}^{\#}$, the intersection of *Blowup*$^{\#}$ with the xy-plane

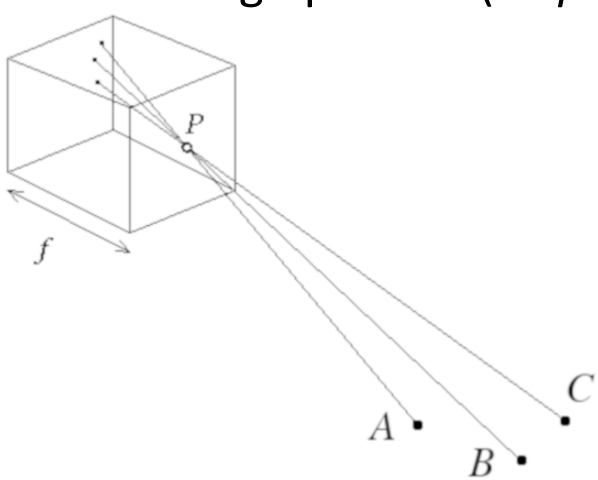


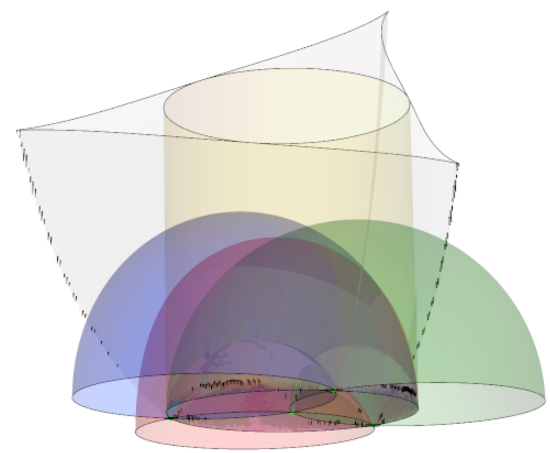


Applying Basic Algebraic Geometry

Each point on the circular boundary of each tiny disc corresponds to a directed line through a vertex (A, B or C), but directed differently on the different *sheets* (copies of the xy-plane), and this suggests how to glue the sheets together

To be specific, let P_0, P_1, P_2 and P_3 be the four sheets, and regard that at a general point p on P_0, the three lines through p are all directed towards their vertices, or equivalently, all directed away from their vertices; and that at a general point p on P_1 (*resp.* P_2, P_3), the line through p and A (*resp.* B, C) is directed differently than the other two lines through p



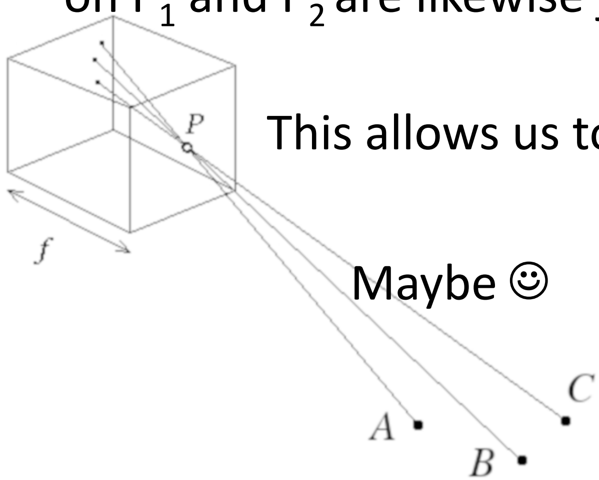


Applying Basic Algebraic Geometry

Near A, after removing the tiny open discs, the boundary circles on P_0 and P_1 are joined, but ***antipodally,*** and likewise the boundary circles on P_2 and P_3 are joined

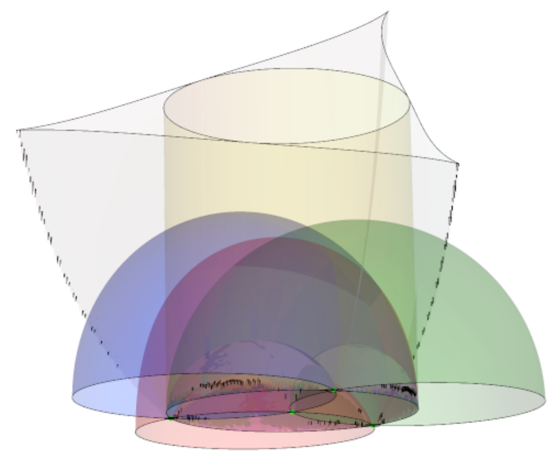
Near B, the boundary circles on P_0 and P_2 are likewise joined, and the boundary circles on P_1 and P_3 are likewise joined

Near C, the boundary circles on P_0 and P_3 are likewise joined, and the boundary circles on P_1 and P_2 are likewise joined



This allows us to somewhat visualize $\mathit{Blowup}_{xy}^{\#}$

Maybe ☺



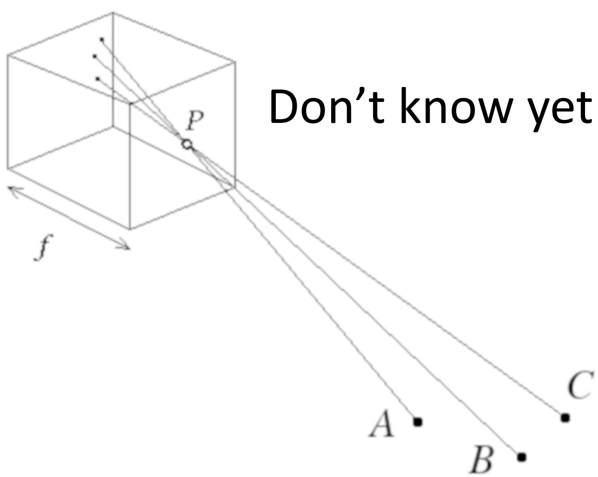
Applying Basic Algebraic Geometry

The four lifts of the circumcircle mentioned earlier weave in between the different sheets, each of these being a circle involving only three out of the four sheets

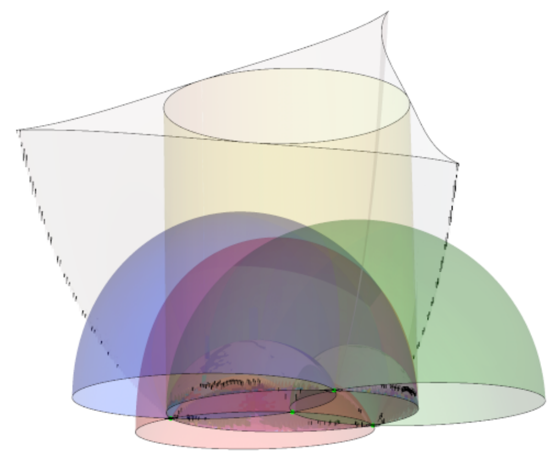
This makes for a somewhat complicated but interesting picture

To obtain $\mathit{Blowup}_{xy}{}^{\dagger}$ we must collapse each of these four circles to a point

Will this surface turn out to be nearly as nice as the simple real torus obtained by Maienschein and Rieck?



Don’t know yet

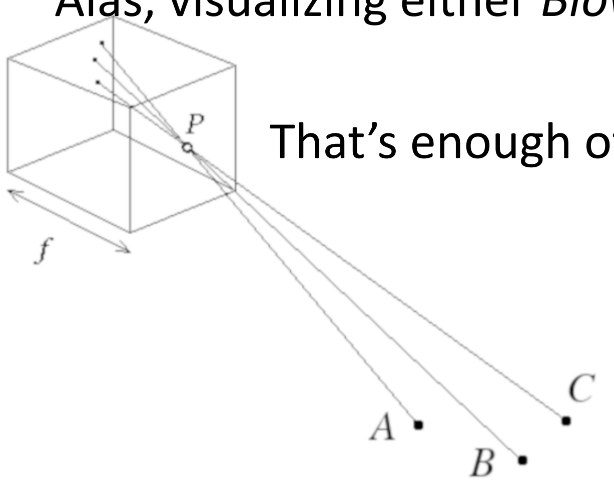


Applying Basic Algebraic Geometry

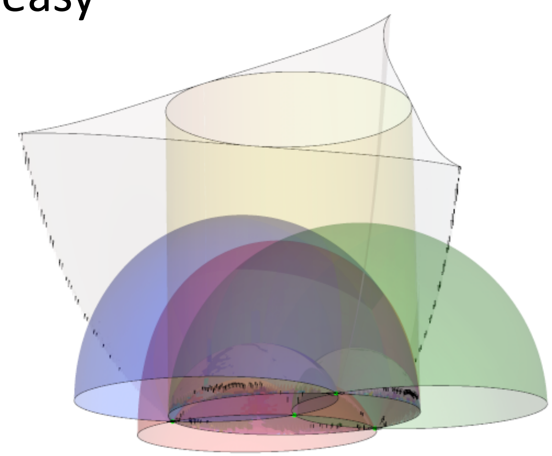
We can also consider the complexified version of *Blowup*† where complex numbers are used instead of real numbers; call this *Blowup*$_{\mathbb{C}}$†

By means of Grunert's system, we still have well-defined local coordinates (c_1, c_2, c_3), and we may still speak of "strongly related" and "weakly related" points in *Blowup*$_{\mathbb{C}}$†

Alas, visualizing either *Blowup*† or *Blowup*$_{\mathbb{C}}$† might not be so easy



That's enough of this for now

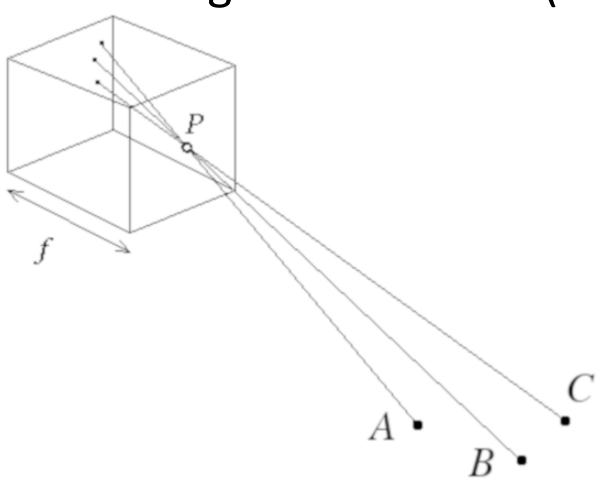


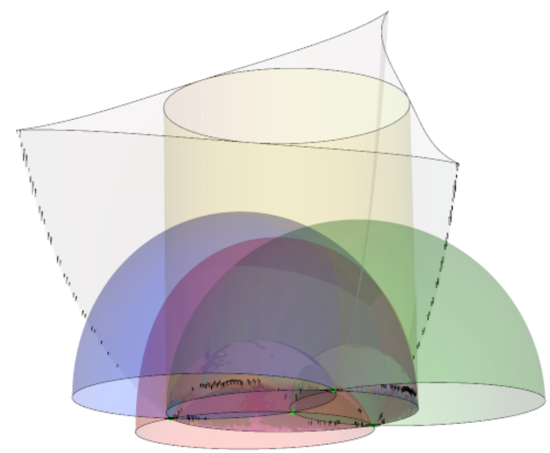
A Family of Space Curves

One tool that has proven to be quite helpful in exploring the features that arise in studying the P3P problem, especially the *CSDC,* are certain curves in xyz-space

Recall the quantities x_L and y_L that were introduced earlier, and which can be expressed as rational functions of x, y and z^2

The set of points that have a specified value of x_L , or a specified value of y_L , form an algebraic surface (variety of dimension two) in xyz-space





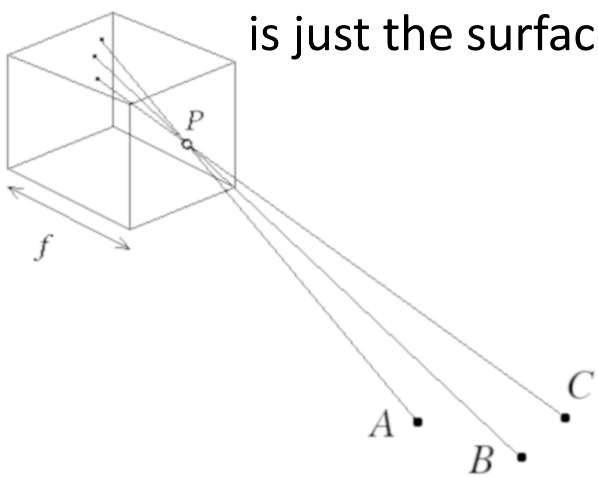
A Family of Space Curves

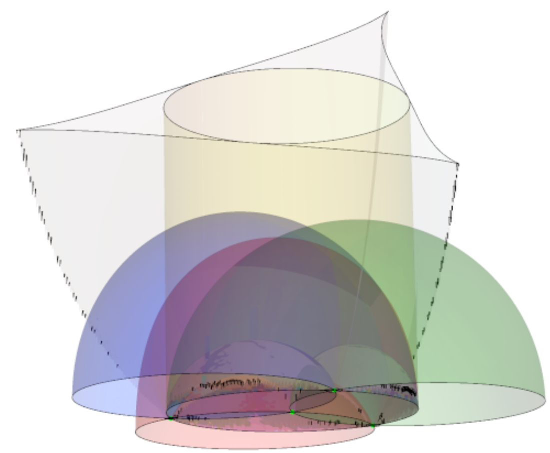
The points that have a specified value of x_L and a specified value of y_L form an algebraic curve (variety of dimension one) in xyz-space

We say "curve" here though this variety need not be connected in xyz-space

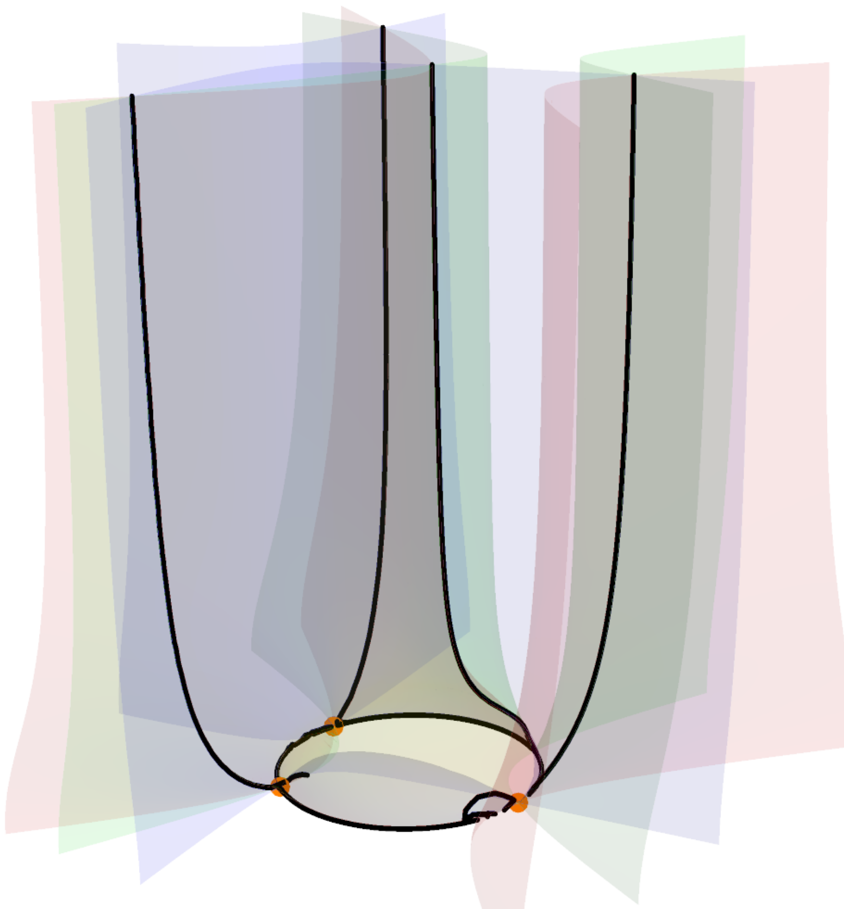
These are the special space curves that have been helpful in analzing P3P

The union of the space curves for which $(9 + 12\, x_L + x_L^2 + y_L^2)^2 = 4\,(3+2x_L)^3$ is just the surface *DC* ∪ *CSDC*





A Family of Space Curves

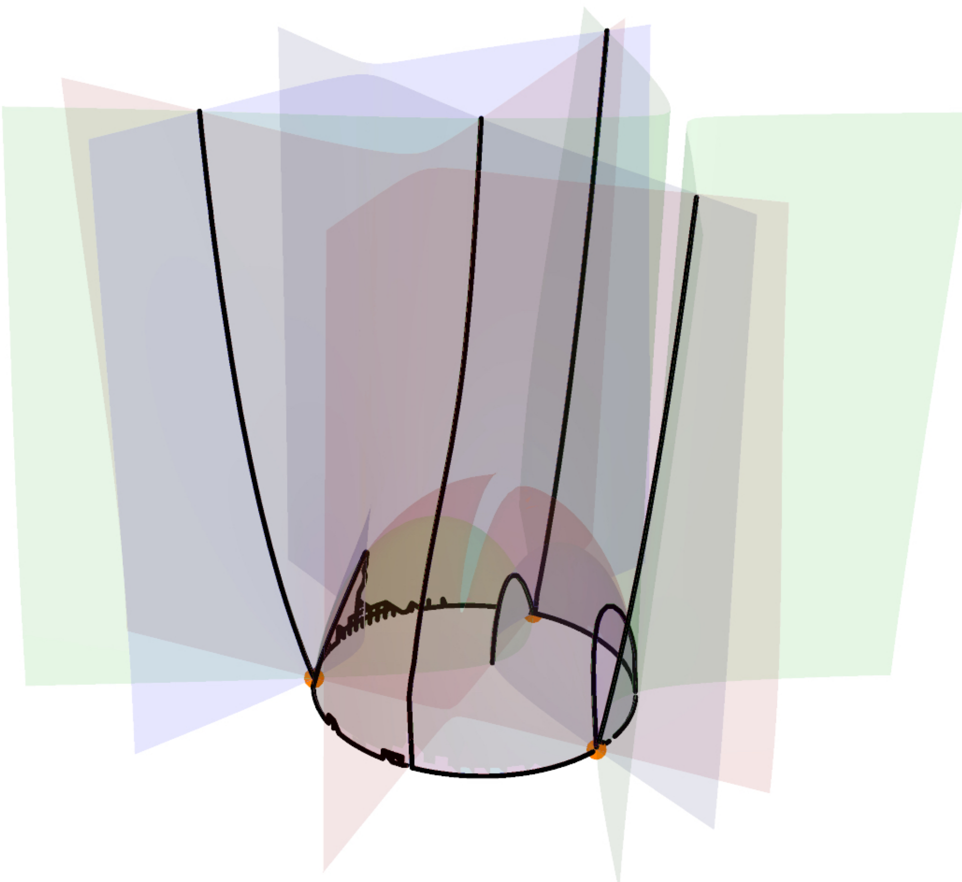


Ignoring the circumcircle, we would be inclined to say that we see several black "curves" in this image

However, though they are not connected, together with their reflections in the xy-plane, these "curves" constitute a single algebraic curve (one-dimensional variety)

The colored surfaces here are contours for certain linear combinations of x_L and y_L that arise naturally

A Family of Space Curves

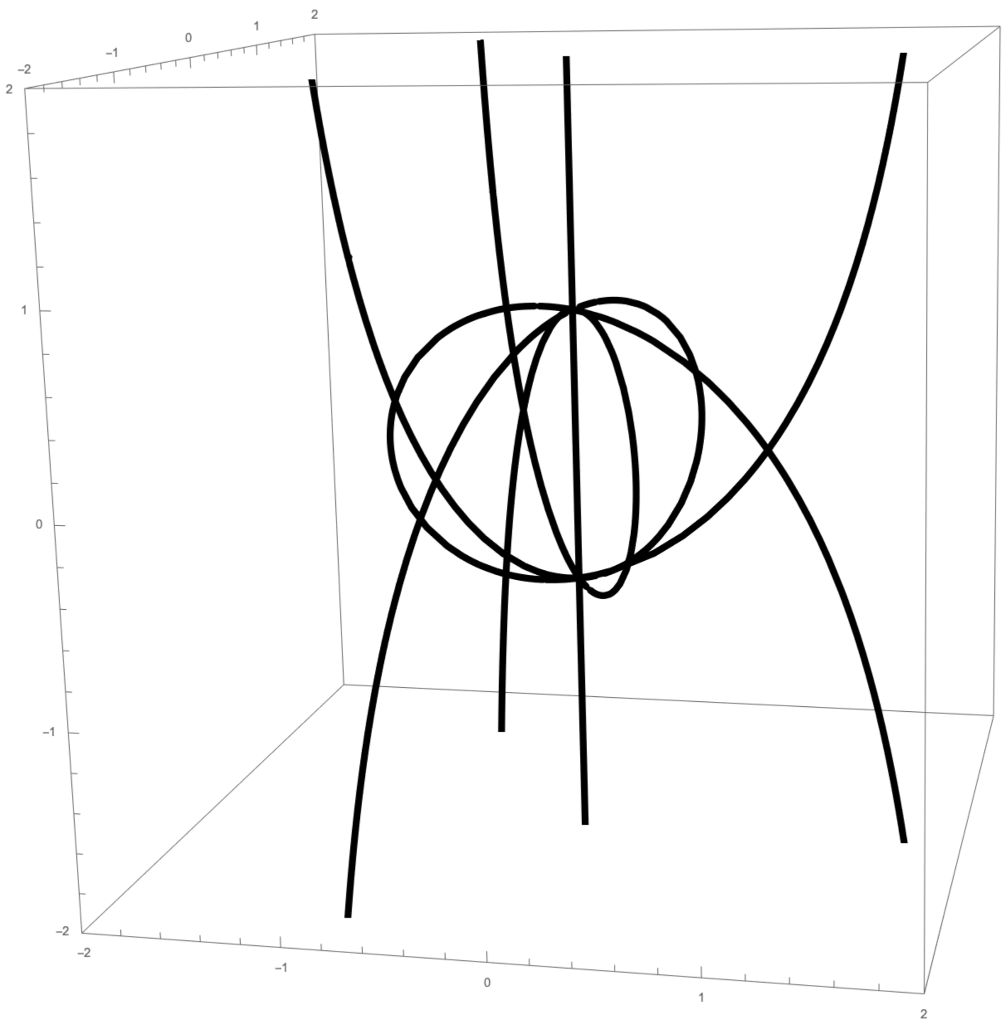


(ignore a rendering issue near circumcircle)

Some claims concerning the intersection of any special space curve and the xy-plane:

1. Can pass through a vertex (control point) from any direction
2. Can go through other points on the circumcircle, but will always be tangent to a tangent plane of the danger cylinder (typically does this one or three times)
3. Always passes through the orthocenter exactly one time and always vertically (different curvatures)
4. No other possible intersections with the xy-plane

A Family of Space Curves



In a sense, the example here is the simplest of all the non-trivial special space curves

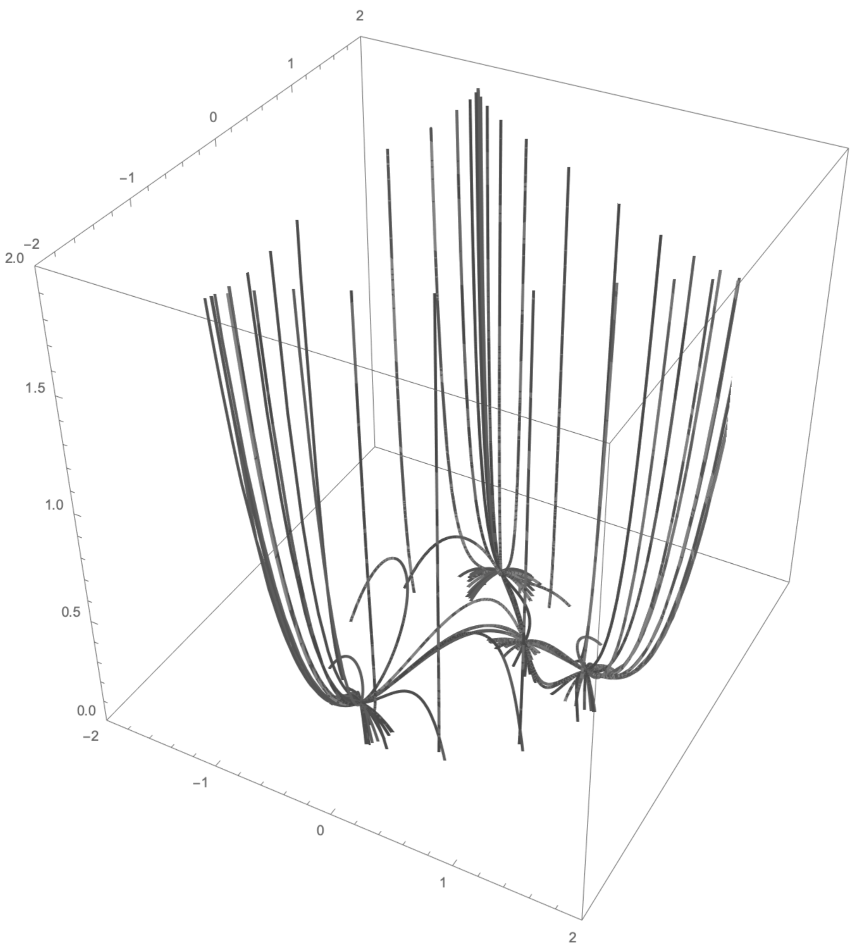
It is for the equilateral triangle case, *i.e.* when $x_H = y_H = 0$

It also uses parameter values $x_L = y_L = 0$

There are four connected components for this curve, including the z-axis

The limiting points as |z| goes to infinity project onto these points in the xy-plane: (0, 0), (2, 0) , (-1, √3) , (-1, -√3)

A Family of Space Curves



This figure gives some sense of the claim that *CS* ∪ *CSDC* can be decomposed as a disjoint union of some of the special space curves

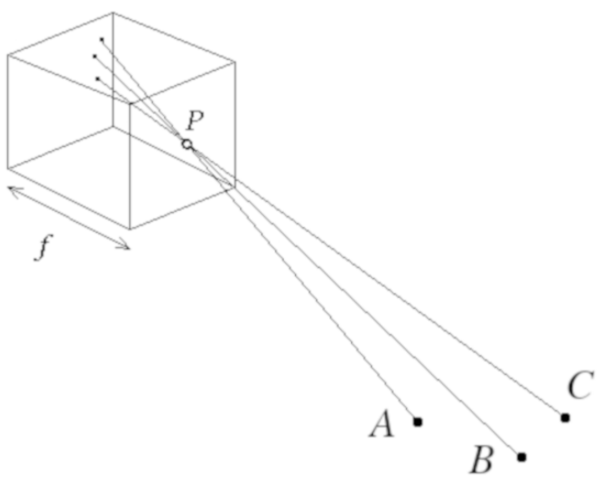
The DC portion of these curves are just vertical lines along the DC, but these are not shown in this figure

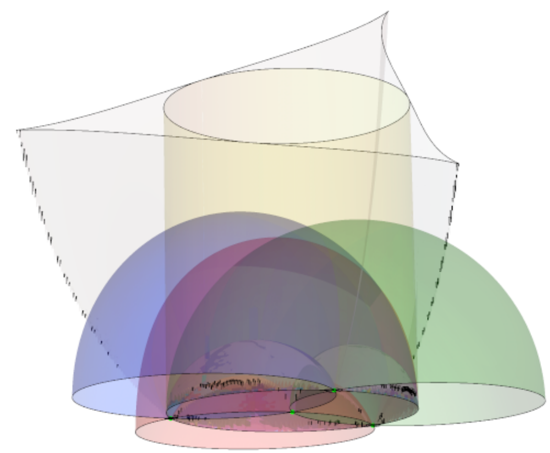
A Family of Space Curves

Alright, the points along any of the special space curves share the same values of x_L and y_L , but what does this mean geometrically; how do we characterize these points geometrically?

It is not yet clear

Now, the projections of the special space curves onto the xy-plane are quite interesting and straightforward to describe



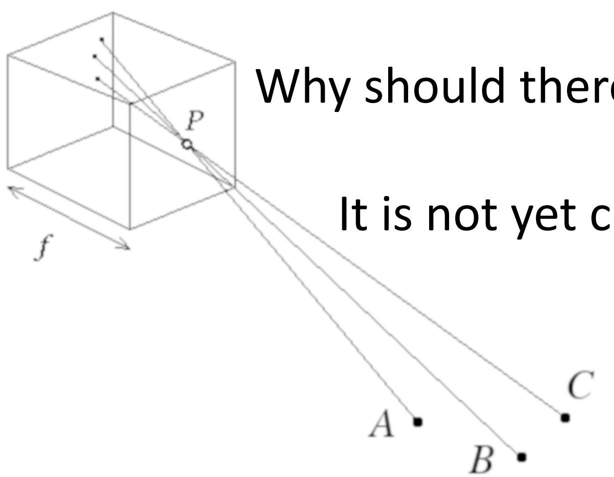


A Family of Space Curves

It turns out that the natural (algebraic) extension of each such planar curve is given by a cubic polynomial in x and y, whose coefficients are simple polynomials in x_H, y_H, x_L and y_L

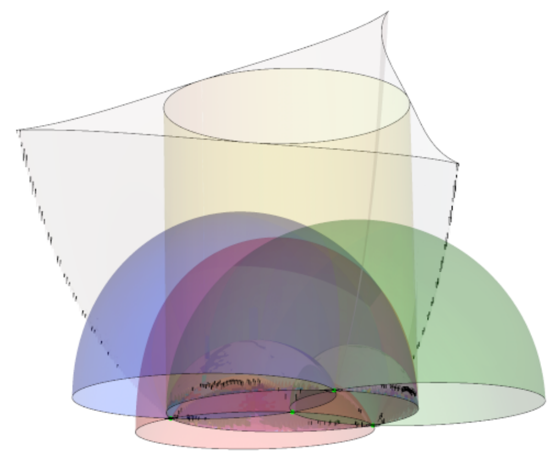
These cubic curves pass through each of the three triangle vertices (A, B, C) and also through the triangle's orthocenter H; we call these the *circum-cubic curves*

They also have asymptotes that are parallel to the triangle's altitude lines

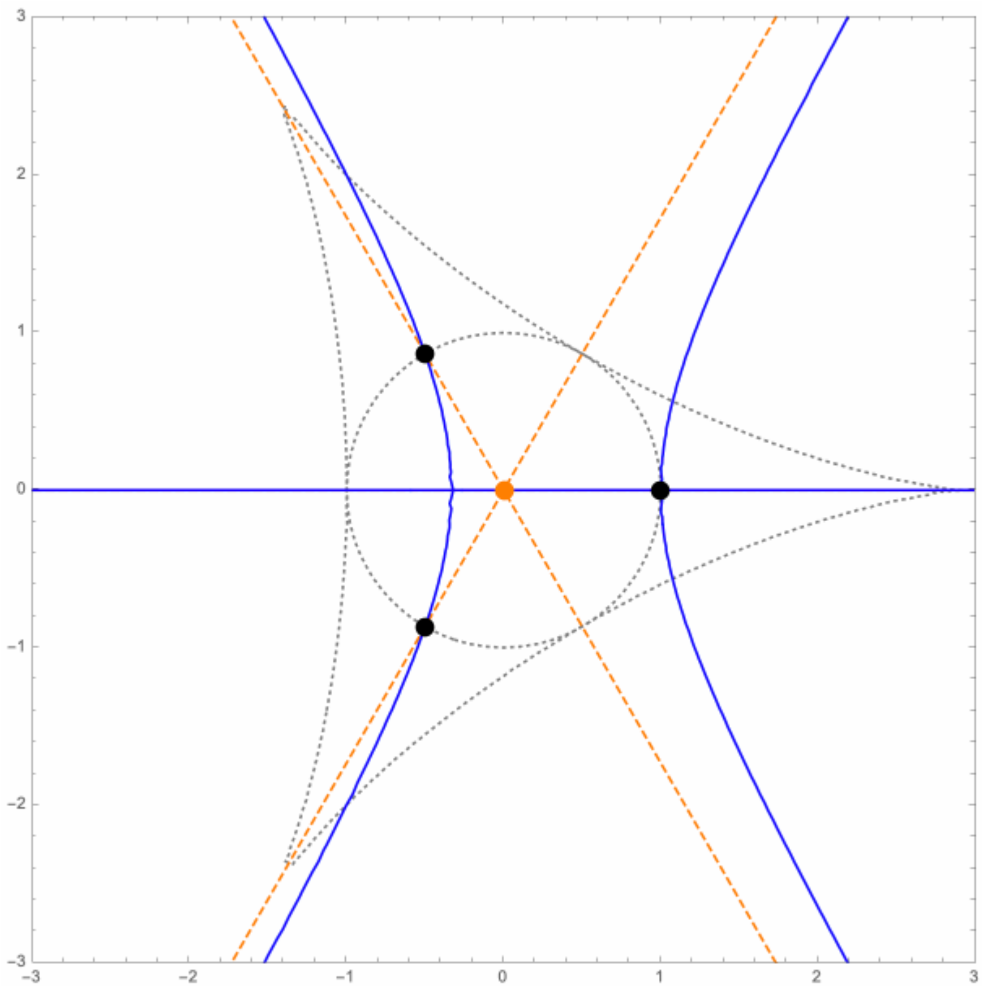


Why should there things be so?

It is not yet clear



A Family of Space Curves



Various Examples of the Circum-cubic Curves for Various Control Point Positions

(Wait for animation)

The blue curve is the natural extension of the projection of one of the special space curves onto the xy-plane (the plane containing A, B and C)

This curve is given by a third-degree polynomial in x any y

The curve always passes through the triangle's vertices (black dots), its orthocenter (orange dot), and three points at infinity corresponding to the directions of the triangle's altitude lines (orange dashed lines)

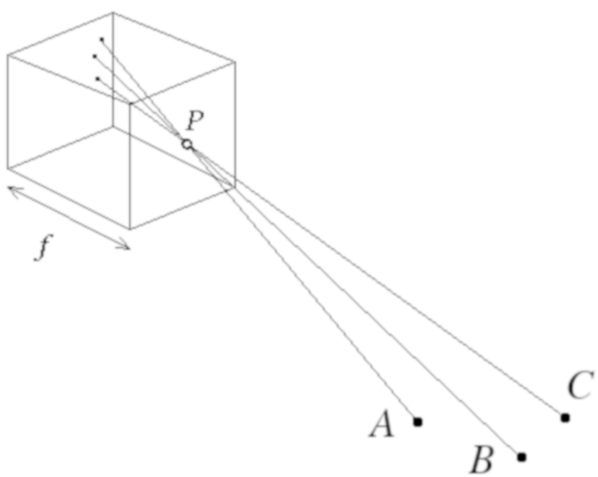
This leaves two degrees of freedom in the possible curves, corresponding to the choices of x_L and y_L

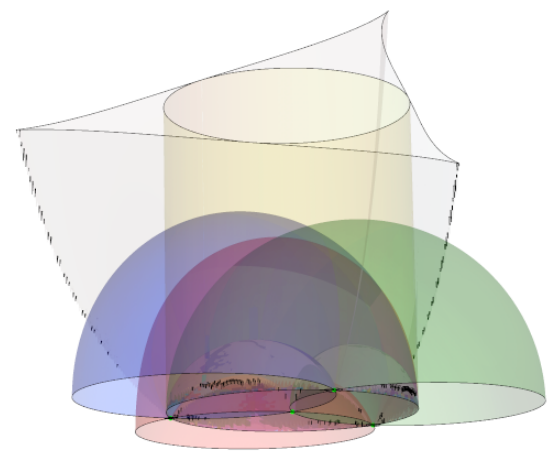
A Family of Space Curves

It has also been helpful to realize the special space curves as the integral curves of a vector field that is well behaved outside of the unit sphere, but rather irratic inside the unit sphere

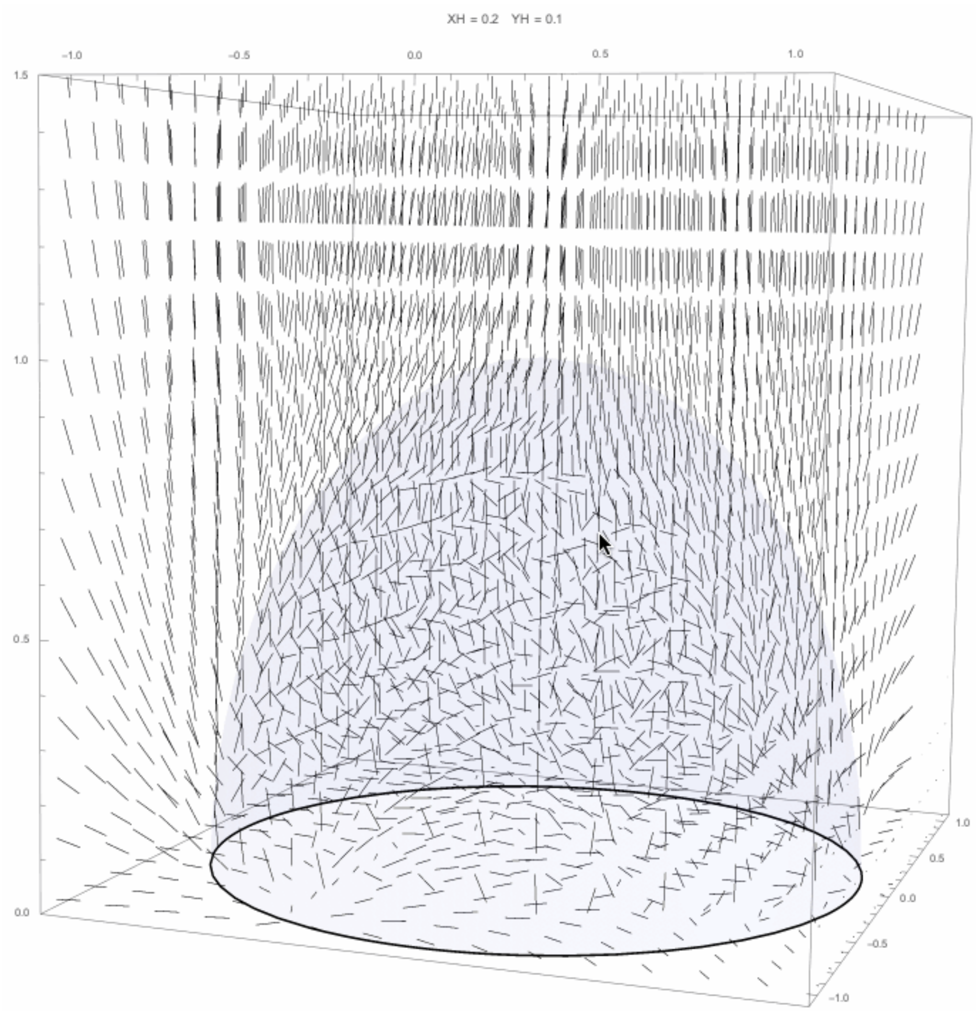
The next slide is an animation that shows the tangents for this vector field, but not the magnitudes and directions, which are irrelevant

This is followed by a slide that shows what the vector field looks like in the xy-plane





A Family of Space Curves



A normalized vector field

This vector field is obtained by normalizing the cross product of the gradient fields of x_L and y_L

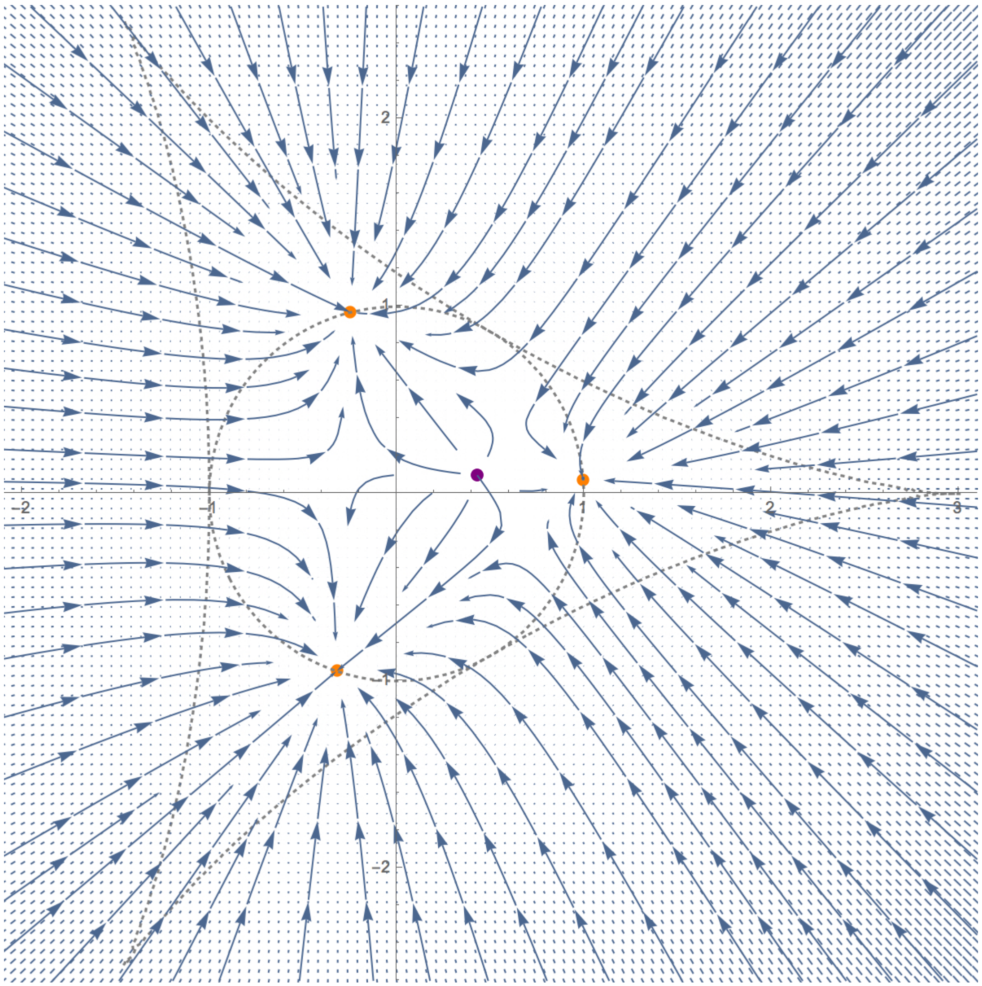
The integral curves of this field are the special space curves on which x_L and y_L are constant

The sphere for which the circumcircle of ΔABC is a diameter (*i.e.* the unit sphere in the xyz-coordinate system) is such that the vector field is quite well behaved outside this sphere

Not so inside the sphere!

We call the sphere the *turbulence sphere* due to the irratic behavior of the vector field and the special space curves inside the sphere

A Family of Space Curves



The vector field in the xy-plane

The orange dots are the control points (triangle vertices)

The purple dot is the orthocenter

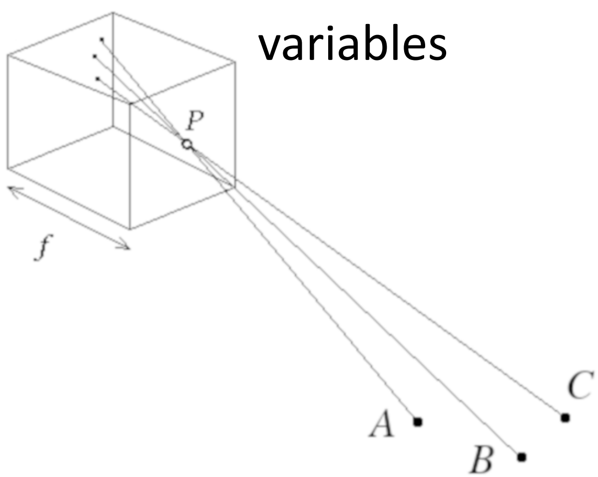
A Curious Family of Birational Transformations

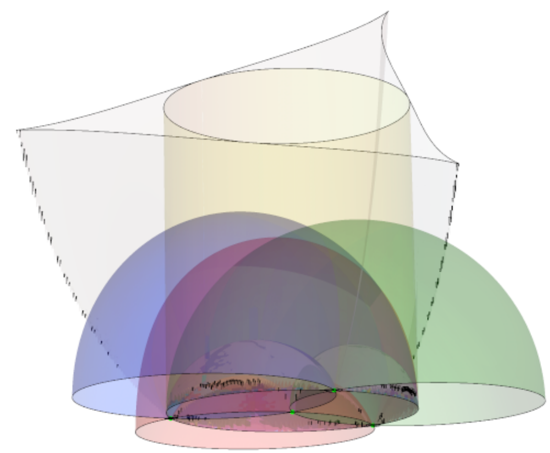
While working to simplify the formulas for x_L and y_L , a certain family (parameterized by x_H and y_H) of *birational transformation* popped up that not only simplifies these formulas, but that also have some interesting geometric properties

They are transformations between x, y and z^2, and quantities μ, ν and ξ ,

$$(x, y, z^2) \quad \longleftrightarrow \quad (\mu, \nu, \xi)$$

where either set of variables can be expressed rationally in terms of the other set of variables

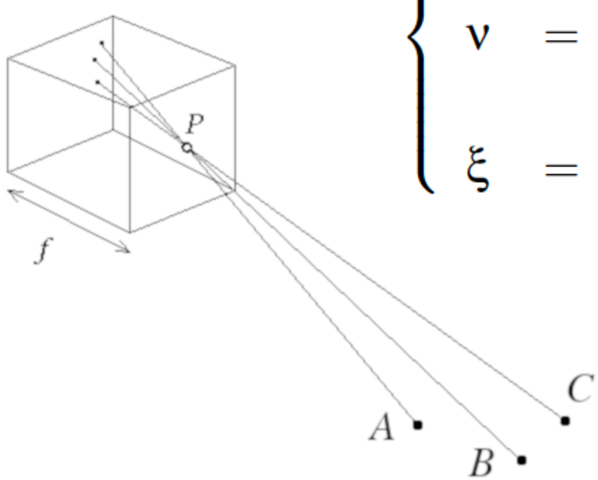




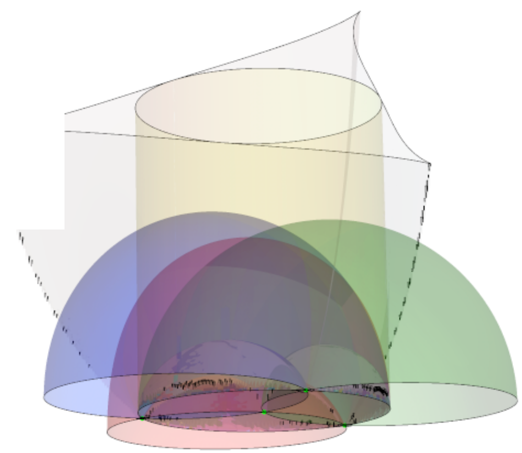
A Curious Family of Birational Transformations

$$\begin{cases} x &= (\mu + X_H \xi) \;/\; (1+\xi) \\ y &= (\nu + Y_H \xi) \;/\; (1+\xi) \\ z^2 &= (\xi - 1)(1 - x^2 - y^2) \;/\; 2\xi \end{cases}$$

$$\begin{cases} \mu &= [\, 2x\,(x^2+y^2+z^2-1) + X_H(1-x^2-y^2)\,] \\ & \quad / \,(\, x^2+y^2+2z^2-1 \,) \\ \nu &= [\, 2y\,(x^2+y^2+z^2-1) + Y_H(1-x^2-y^2)\,] \\ & \quad / \,(x^2+y^2+2z^2-1) \\ \xi &= (x^2+y^2-1) \;/\; (x^2+y^2+2z^2-1) \end{cases}$$



(from Rieck 2018; the capital X and Y here should be lowercase)



A Curious Family of Birational Transformations

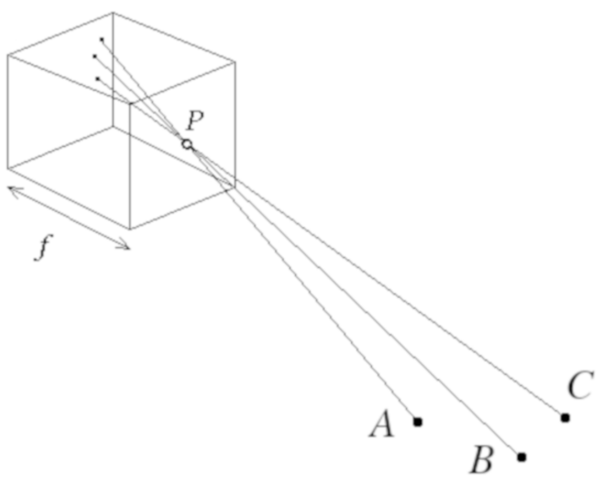
Note that $(x^2+y^2+z^2-1) / z^2 = (1+\xi) / (1-\xi)$ and $(x^2+y^2-1) / z^2 = 2\xi / (1-\xi)$

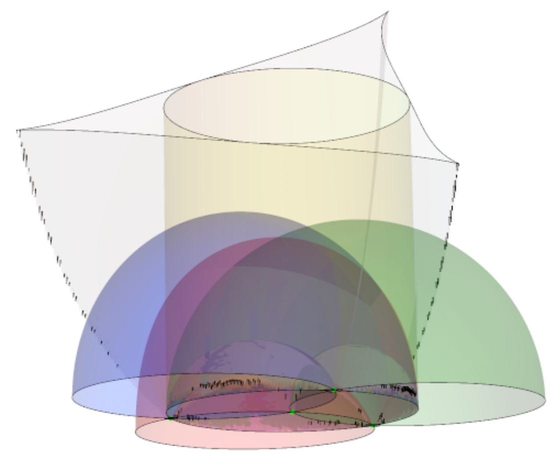
Consequently:

$|\xi| > 1 \quad \Rightarrow \quad$ (x,y,z) is inside the unit sphere

$-1 < \xi < 0 \quad \Rightarrow \quad$ (x,y,z) is outside the unit sphere but inside the danger cylinder

$0 < \xi < 1 \quad \Rightarrow \quad$ (x,y,z) is outside the danger cylinder



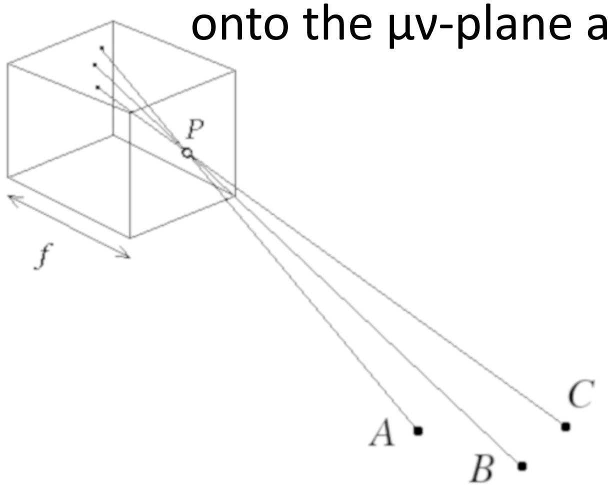


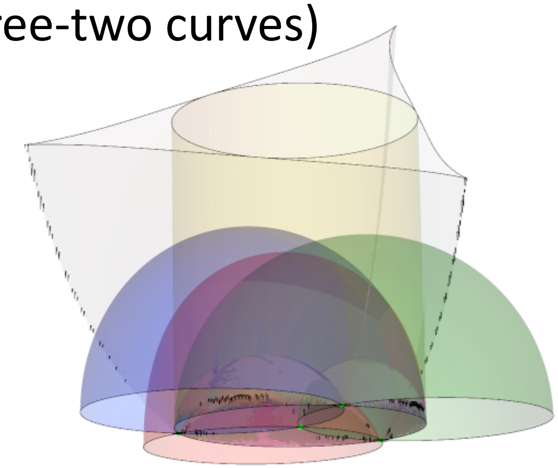
A Curious Family of Birational Transformations

The simpler formulas for x_L and y_L are just these:

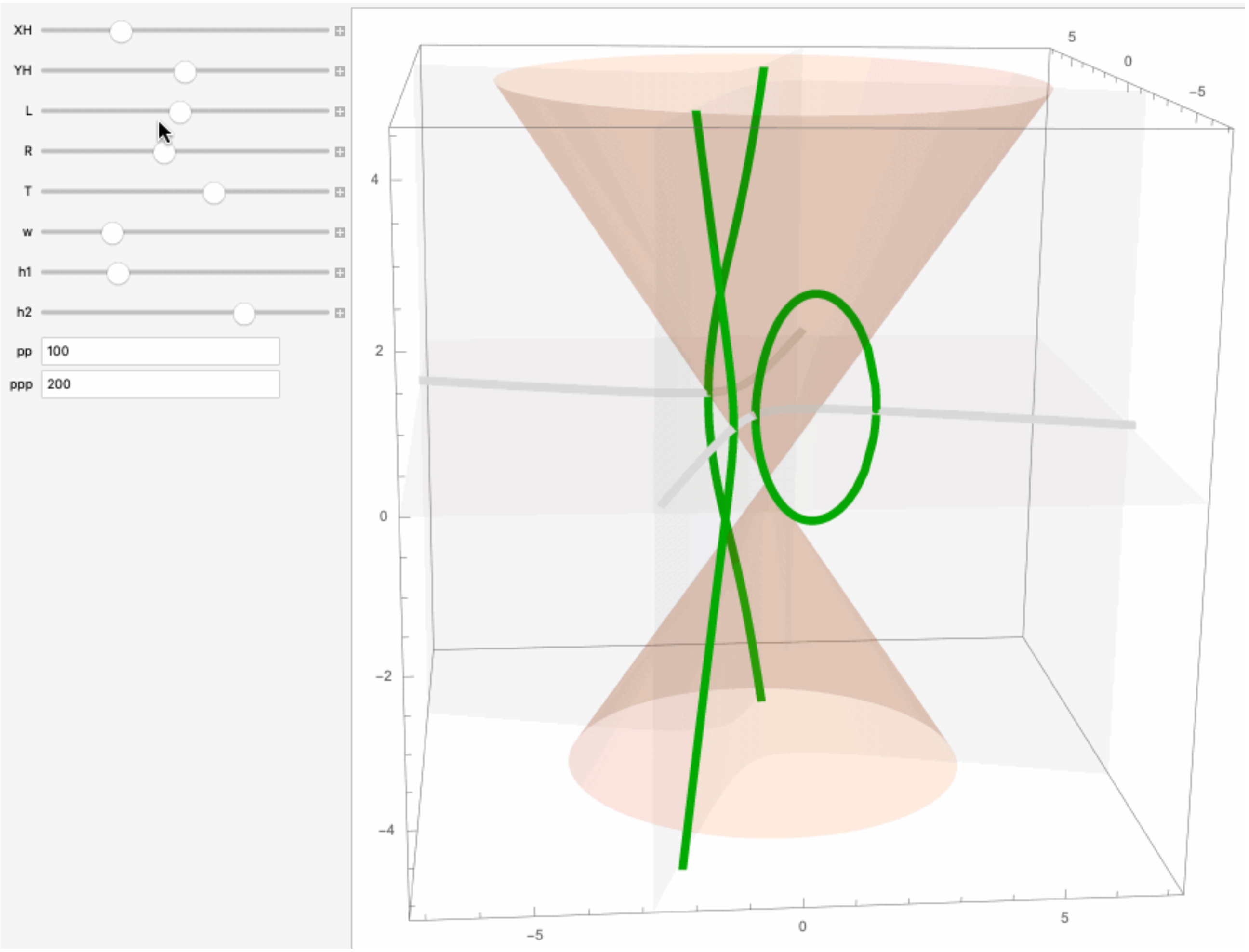
$$x_L = [\ \mu^2 - 2\mu - \nu^2 - (x_H^2 - 2x_H - 2y_H)\,\xi^2\ ]\ /\ [\ 1 - \xi^2\ ]$$
$$y_L = [\ 2(\mu + 1)\,\nu - 2(x_H + 1)\,y_H\,\xi^2\ ]\ /\ [\ 1 - \xi^2\ ]$$

While that is somewhat interesting, more interesting perhaps is the fact that the special curves in xyz-space transform to curves in μνξ-space whose projections onto the μν-plane are just rectangular hyperbolas (degree-two curves)





A Curious Family of Birational Transformations



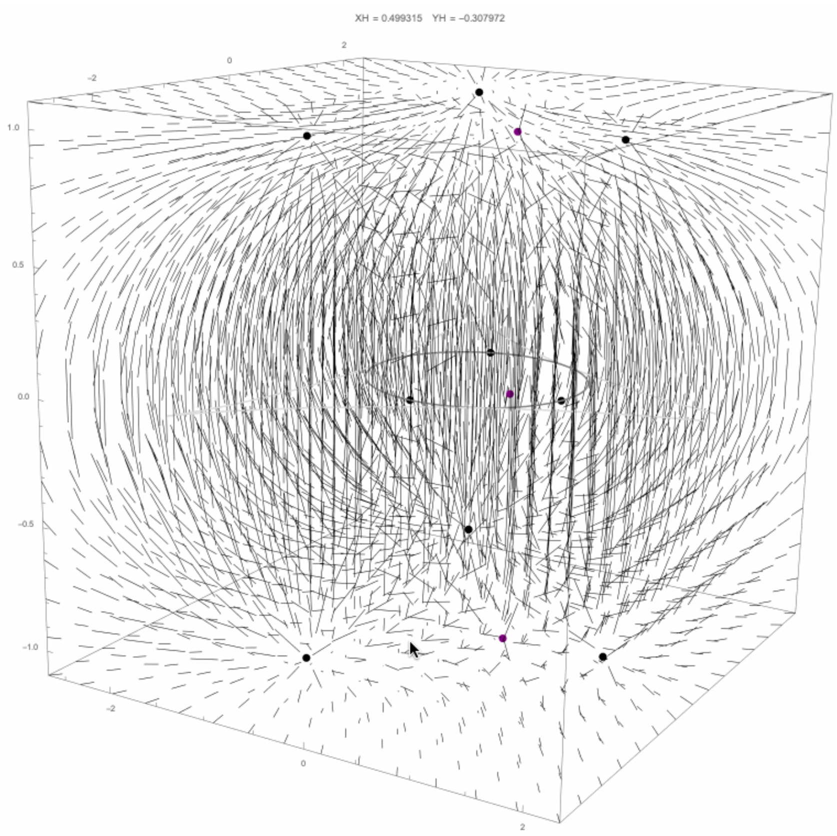
The special space curves realized in μνξ-space

(Wait for animation)

These (green) curves in μνξ-space are simpler than the corresponding curves in xyz-space

The projections of these onto the μν-plane are rectangular hyperbolas (gray curves)

A Curious Family of Birational Transformations



The vector field in μνξ-space

(Wait for animation)

The vector field for the special μνξ-space curves is proportional to

$$\Big(\; [(x_H-1)^2-y_H^2](\mu+1) + 2(x_H+1)y_H\nu \;-\; (\mu+1)\,[(\mu-1)^2 + \nu^2]\,,$$
$$2(x_H+1)y_H(\mu-1) + [y_H^2-(x_H-1)^2]\nu - \nu\,[\nu^2 + (\mu-1)\,(\mu+3)]\,,$$
$$(\mu^2+ \nu^2-1)(1-\xi^2) / \xi \;\Big)$$

This vector field vanishes at the points $(\mu, \nu, \xi) =$ $(x_H, y_H, \pm1)$, $(2x_1-x_H, 2y_1-y_H, \pm1)$, $(2x_2-x_H, 2y_2-y_H, \pm1)$ and $(2x_3-x_H, 2y_3-y_H,, \pm1)$

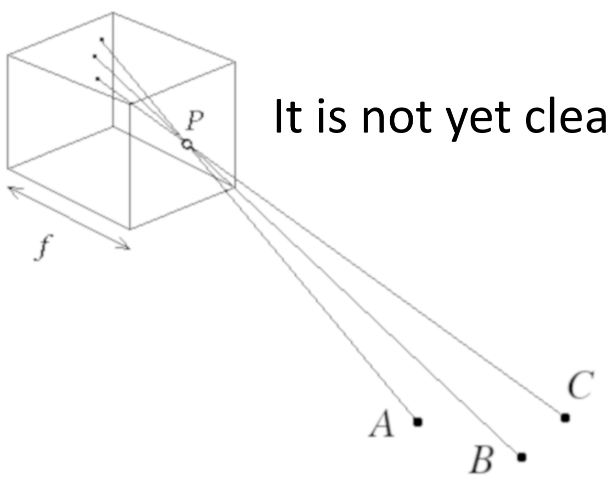
It is generally vertical when $\xi = 0$, and horizontal when $\xi = \pm1$, and when $\mu^2 + \nu^2 = 1$

A Curious Family of Birational Transformations

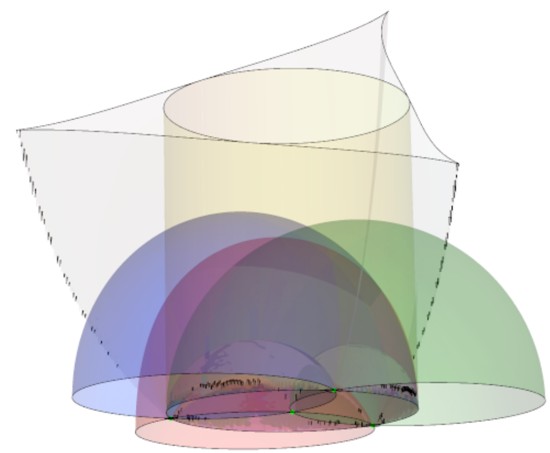
A big surprise, quite a shock really, is that the special double toroids, $DToroid_{A}$, $DToroid_{B}$ and $DToroid_{C}$, are transformed into elliptical double cones in μνξ-space

Also, the circumcircle expands (blows up) into a very similar double cone, while the rest of the danger cylinder collapses (blows down) to a circle

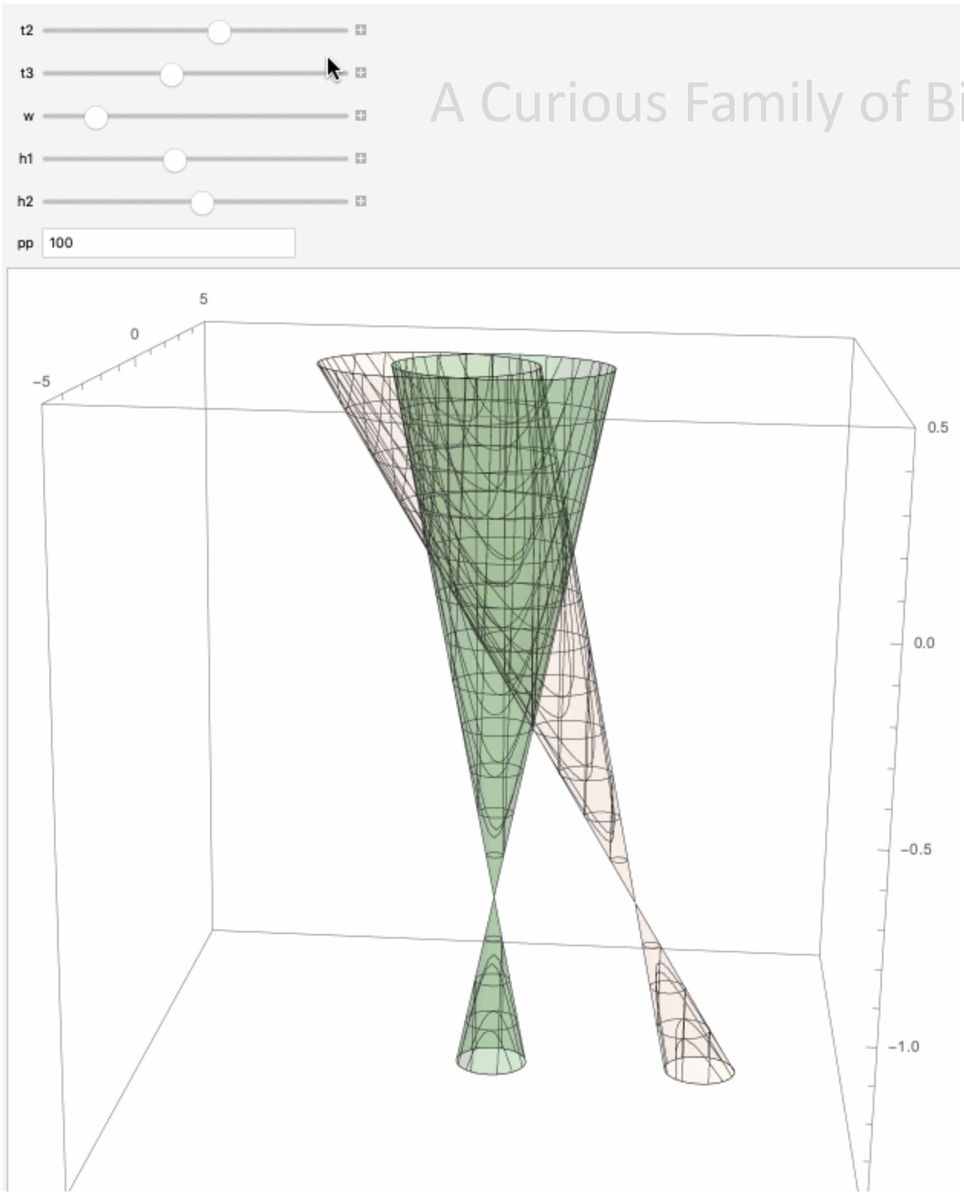
The bilinear transformation arose from an aspect of P3P that did not seem to involve toroids and double toroids, so why are these surfaces so profoundly affected?



It is not yet clear



A Curious Family of Birational Transformations



A couple elliptical double cones in μνξ-space

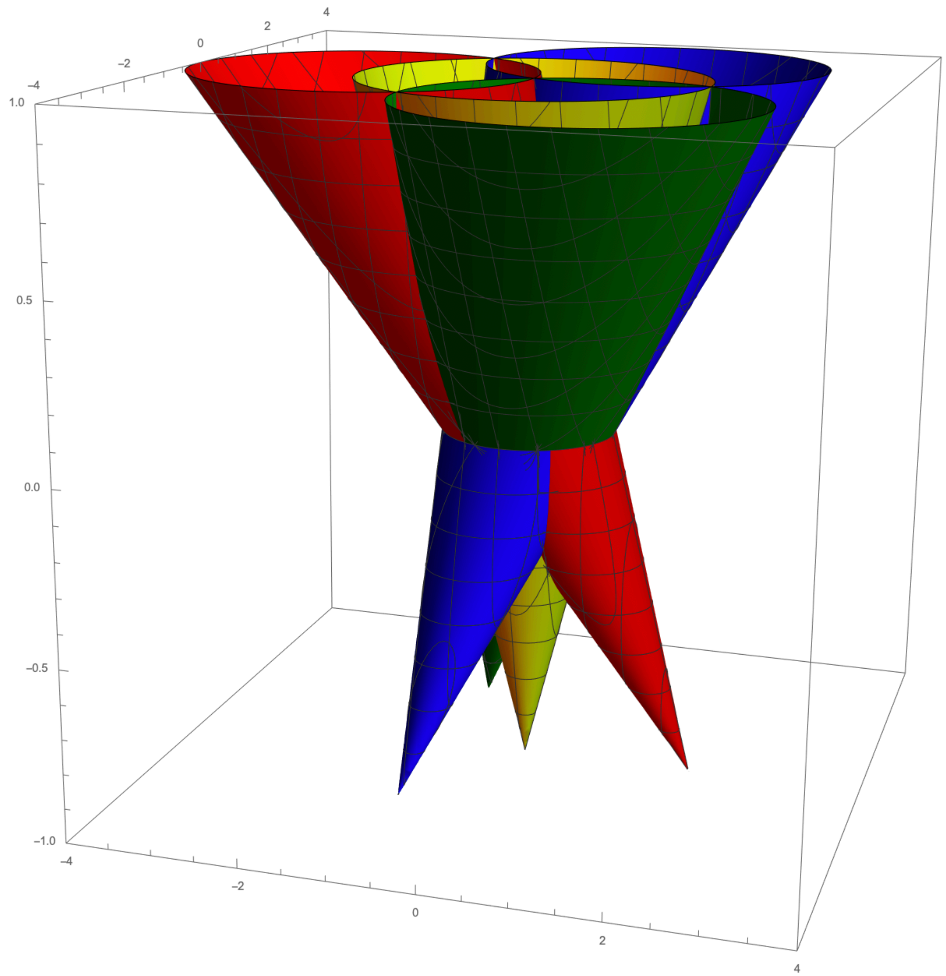
(Wait for animation)

The green double cone is the transformation of the ΔABC's circumcircle (which blows up in a sense)

The other double cone is the transformation of one of the three special double toroids (*i.e.* the rotatation of the circumcircle about one of the triangle ΔABC's sidelines)

In the animation, the underlying triangle ΔABC is occasionally changed

A Curious Family of Birational Transformations

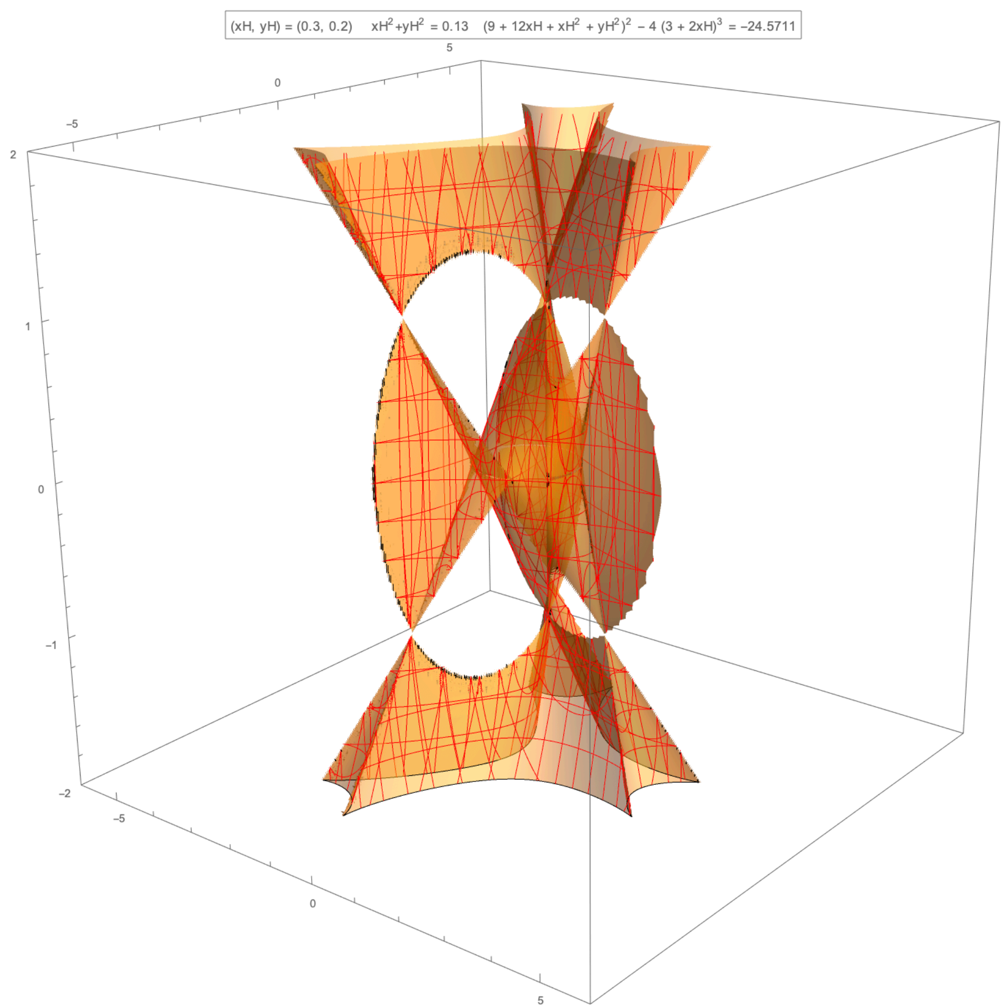


For the equilateral triangle case, here are portions of the elliptical double cones in μνξ-space that correspond to the blowup of the circumcircle (yellow) and the three special double toroids (red, green and blue)

All of these pass through the unit circle $\mu^2 + \nu^2 = 1$ in the μν-plane

All have their apices in the $\xi = -1$ plane

Rather mysterious. 🤔



_CS_ ∪ _CSDC_ in μνξ-space

The formula for *CS* ∪ *CSDC, i.e.* the discriminant for Grunert’s system, also simplifies quite a bit when μνξ-coordinates are used

The formula is invariant under the substitution $\xi \longrightarrow -\xi$ and perhaps there is a geometric reason

Here is what the surface looks like in μνξ-space when $(x_H, y_H) = (0.3, 0.2)$

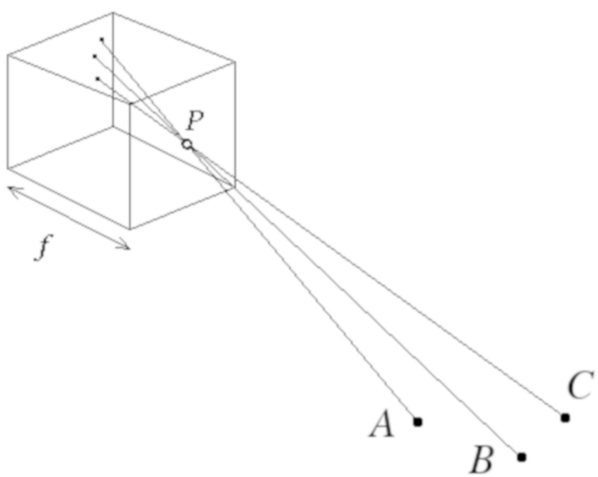
It is of course symmetric about the μν-plane

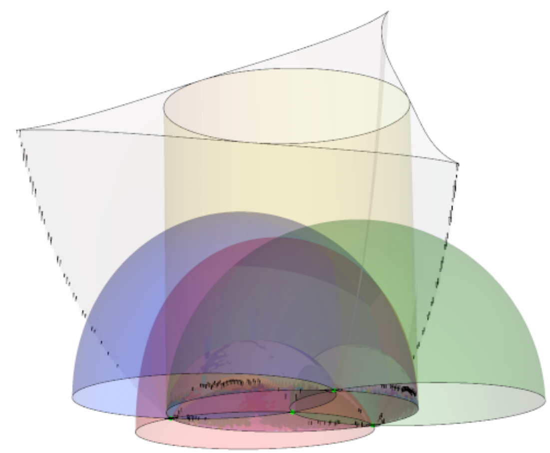
Its intersection with the μν-plane is just the union of the standard deltoid and the unit circle

A Curious Family of Birational Transformations

A theorem in algebraic geometry says that any birational transformation between smooth complex projective varieties can be achieved through a sequence of blowups and blowdowns, though blowups and blowdowns are defined more generally here

Perhaps there is a nice decomposition for the birational transformations being investigated here



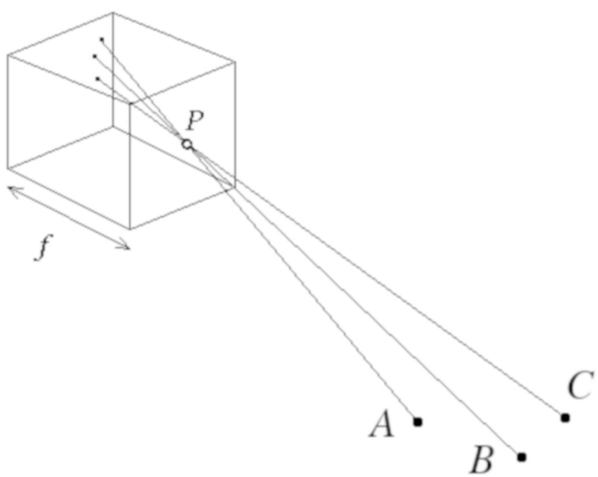


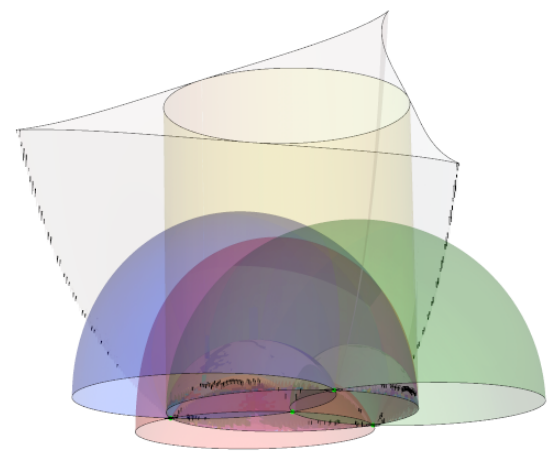
A Curious Family of Birational Transformations

As a step in this direction, perhaps, let us set $W = z^2 / (x^2 + y^2 - 1)$

This yields $x = (\mu + x_H \xi) / (1 + \xi)$, $y = (\nu + y_H \xi) / (1 + \xi)$, $W = (1 - \xi) / 2\xi$,
$\mu = [\, 2(1 + W)\, x - x_H \,] / [1 + 2W]$, $\nu = [\, 2(1 + W)\, y - y_H \,] / [1 + 2W]$, $\xi = 1 / [1 + 2W]$

The birational transformation $(x, y, W) \longleftrightarrow (\mu, \nu, \xi)$ seems pretty simple, and perhaps it has a nice geometric interpretation



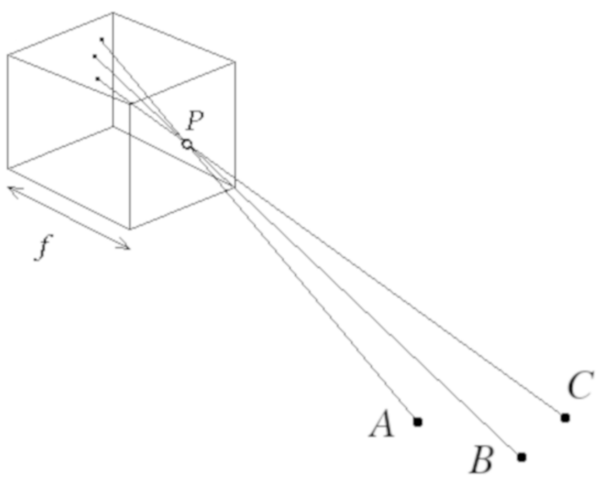


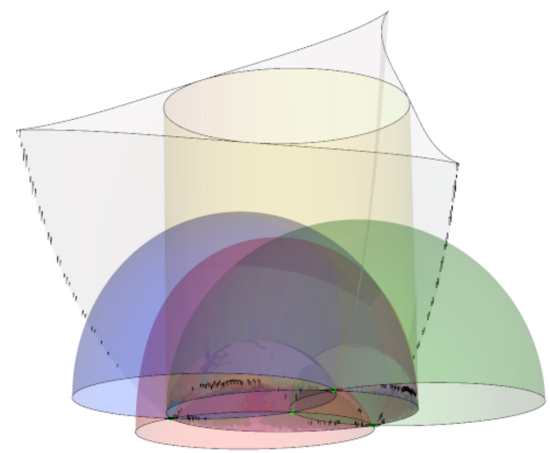
A Curious Family of Birational Transformations

As an alternative to W, we could use instead $Q = W / (1+W) = z^2 / (x^2 + y^2 + z^2 - 1)$

This yields $x = (\mu + x_H \xi) / (1 + \xi)$, $y = (\nu + y_H \xi) / (1 + \xi)$, $Q = (1 - \xi) / (1 + \xi)$,
$\mu = [\, 2x - x_H (1\text{-}Q) \,] / (1+Q)$, $\nu = [\, 2y - y_H (1\text{-}Q) \,] / (1+Q)$, $\xi = (1\text{-}Q) / (1+Q)$

Letting $\mu' = \mu + x_H \xi$ and $\nu' = \nu + y_H \xi$, we have $x = \mu' / (1 + \xi)$, $y = \nu' / (1 + \xi)$,
$Q = (1\text{-}\xi) / (1+\xi)$, $\mu' = 2x / (1+Q)$, $\nu' = 2y / (1+Q)$, $\xi = (1\text{-}Q) / (1+Q)$



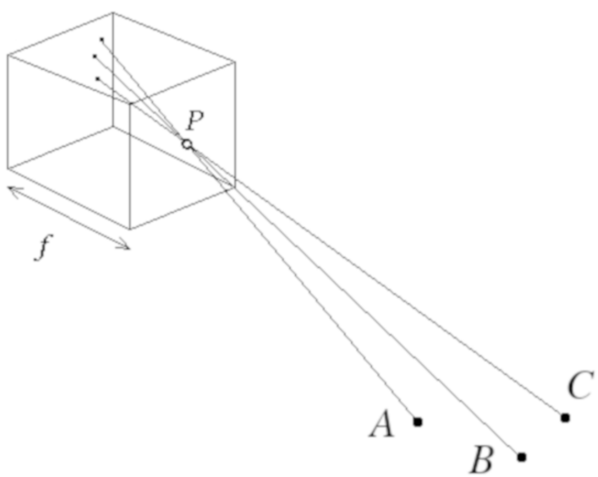


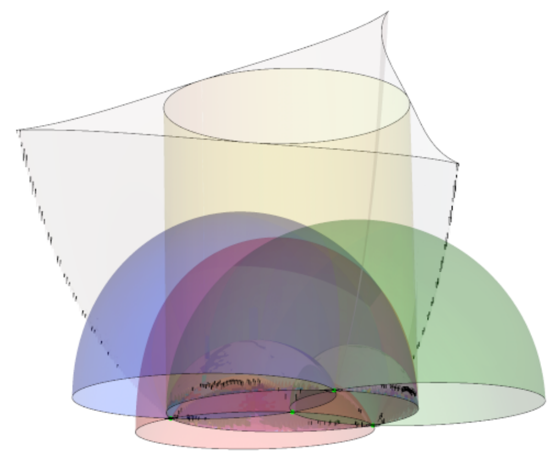
A Curious Family of Birational Transformations

The birational transformation $(x, y, Q) \longleftrightarrow (\mu', \nu', \xi)$ seems especially simple

Still, a good geometric sense of all this is pretty much lacking at the moment, and certain phenomena are currently mysterious

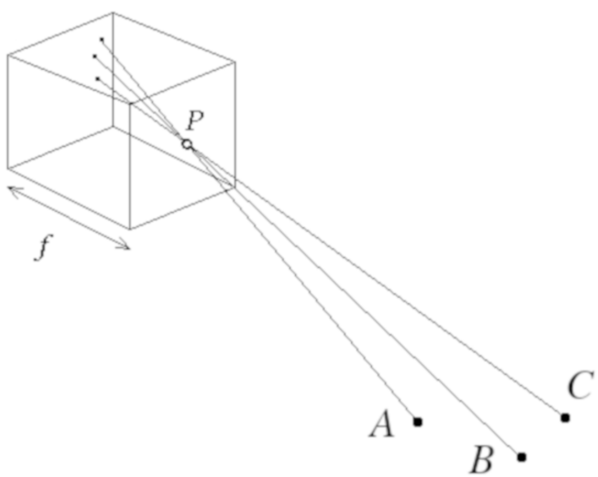
It has been observed for instance that any torus or double toroid obtained by (3D) rotating the unit circle in the xy-plane about any line in the xy-plane, transforms under $(x, y, z) \longrightarrow (\mu, \nu, \xi)$ to an elliptical double cone -- no idea why

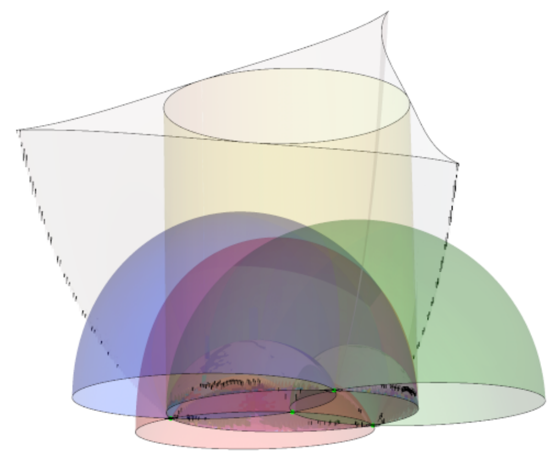




Some Closing Questions

1. Why does a deltoid curve occur in the limiting case as $|z|$ goes to infinity?
2. Why does the deltoid curve formula occur in the "nice" way of writing the otherwise messy discriminant formula for Grunert's system of equations?
3. Is there an intuitive geometric significance to the special xyz-space curves?
4. Why do these curves project onto circum-cubic curves with special properties?





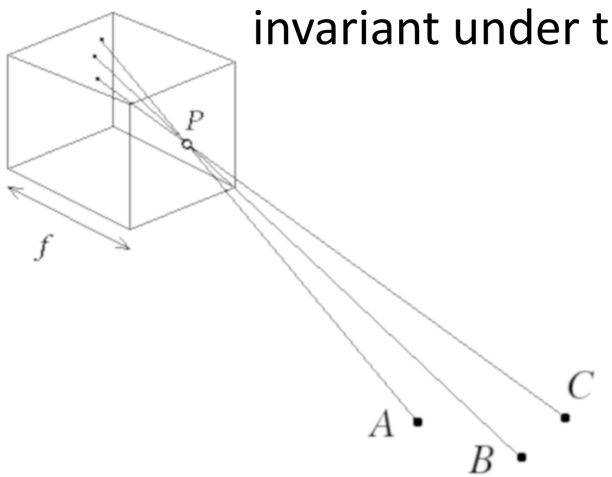
Some Closing Questions

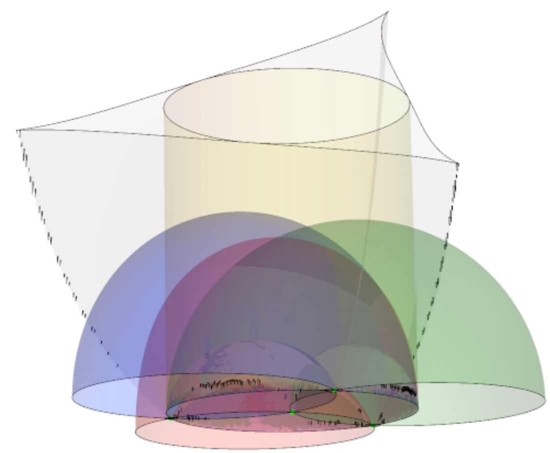
5. Is there an intuitive geometric significance for the transformation $(x, y, z^2) \longleftrightarrow (\mu, \nu, \xi)$?

6. Why do the special μνξ-space curves project onto rectangular hyperbolas?

7. Why do toroids and toruses generated by the unit circle in xyz-space transform into elliptic cones in μνξ-space?

8. Why is the equation for *CS* ∪ *CSDC,* when expressed in μνξ-coordinates, invariant under the substitution $\xi \longrightarrow -\xi$?





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Website: http://mqrieck.com/P3P/P3P_Solutions.html

Thank You

Questions?