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Four Problems in Solid Geometry with Proposed Solutions

Fix three non-coplanar rays from the origin in Cartesian 3-space, $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$. Let α be the angle between $\vec{r}_2$ and $\vec{r}_3$, let β be the angle between $\vec{r}_3$ and $\vec{r}_1$, and let γ be the angle between $\vec{r}_1$ and $\vec{r}_2$. The angles α, β and γ are each strictly between 0 and π (radians), and satisfy these four evident inequalities: $\alpha+\beta+\gamma < 2\pi$, $\alpha < \beta+\gamma$, $\beta < \gamma+\alpha$ and $\gamma < \alpha+\beta$. Now, extend the three rays to obtain three lines, l_1 , l_2 and l_3 . Also fix three angles $\angle A$, $\angle B$ and $\angle C$, each strictly between 0 and π, whose sum is π. Consider the following two related problems:

The **Lines Problem**: Do there exist points *A, B, C* on l_1 , l_2 , l_3 , respectively, such that the interior angles of the triangle *ABC* are $\angle CAB = \angle A$, $\angle ABC = \angle B$, $\angle BCA = \angle C$?

The **Rays Problem**: The same question, but further restrict *A, B, C* to be on the rays $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, respectively.

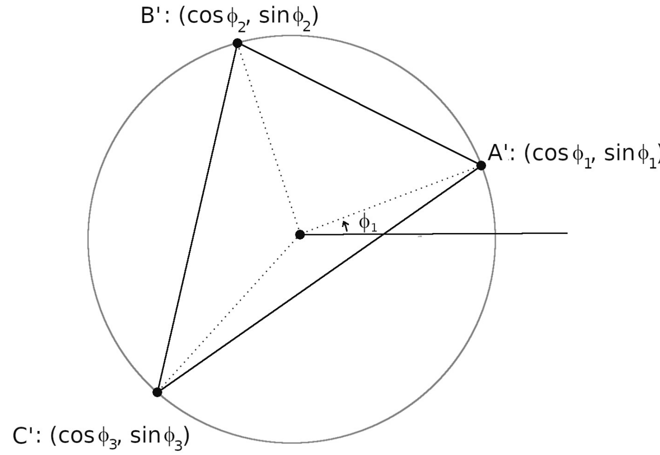
For each of these problems, we seek a system of relationships between the six quantities $\angle A$, $\angle B$, $\angle C$, α , β and γ that provides necessary and sufficient conditions for answering the question. It turns out that these problems lead to quite different solutions depending on whether the triangle *ABC* is to be acute or obtuse. (Right triangles will be ignored here but can be dealt with as continuous limits in the acute triangle case.) Accordingly, let us divide the original two problems into four problems as follows:

LA: the Lines Problem when $\angle A$, $\angle B$, $\angle C$ are all less that π/2.
LO: the Lines Problem when one of $\angle A$, $\angle B$, $\angle C$ is greater than π/2.
RA: the Rays Problem when $\angle A$, $\angle B$, $\angle C$ are all less that π/2.
RO: the Rays Problem when one of $\angle A$, $\angle B$, $\angle C$ is greater than π/2.

By symmetry, for **LO** and **RO**, there is no loss in generality in assuming that $\angle A$ is the obtuse angle. The **LA** problem has a trivial solution: there are no restrictions concerning the six quantities beyond those mentioned above. For each of the other three problems, a system is presented below that extensive experiments (using Mathematica and C++) strongly suggest solves the problem. Each such system is a logical combination of inequalities involving the six quantities. The system for **RA** has been proven to be necessary in order to obtain an affirmative answer, though it has not yet been proven to be sufficient. The systems presented for **LO** and **RO**, though based on considerable analysis, have not yet been rigorously established. Some preliminaries are needed before specifying the proposed system for each of the problems. Throughout, it will be helpful to let $c_1 = \cos\alpha$, $c_2 = \cos\beta$, $c_3 = \cos\gamma$, $C_1 = c_1^2$, $C_2 = c_2^2$, $C_3 = c_3^2$ and $C_0 = c_1 c_2 c_3$.

1. Consider a triangle *A'B'C'* in the Cartesian plane, having its vertices on the unit circle, and such that $\angle C'A'B' = \angle A$, $\angle A'B'C' = \angle B$, and $\angle B'C'A' = \angle C$. Such a triangle is guaranteed to exist. Rotating this triangle about the origin does not affect these properties, and we will assume that it is rotated in order to achieve the following additional property: Denote the coordinates of *A'*, *B'* and *C'* by (x_1, y_1) , (x_2, y_2) and (x_3, y_3), respectively, with $x_j = \cos\phi_j$ and $y_j = \sin\phi_j$ for angles ϕ_j (j = 1,2,3). The extra requirement

is that $\phi_1 + \phi_2 + \phi_3 = 0$. This is always possible, in several ways, in fact, and which of these is used is unimportant. Now define the following quantities:



$$L = 2\,[\,(y_1 + y_2 + y_3)(1 - C_0) + (1 - x_1)\,y_1\,(C_1 - 1) + (1 - x_2)\,y_2\,(C_2 - 1) + (1 - x_3)\,y_3\,(C_3 - 1)\,],$$

$$R = 2\,[\,(1 + x_1 + x_2 + x_3)(1 - C_0) + (1 + x_1)\,x_1\,(C_1 - 1) + (1 + x_2)\,x_2\,(C_2 - 1) + (1 + x_3)\,x_3\,(C_3 - 1)\,],$$

$$H = 1 + 2\,C_0 - C_1 - C_2 - C_3, \qquad K = -H - R,$$

$$D = 4\,H\,(2\,K + 3\,H)^3 - (K^2 + L^2 + 12\,H\,K + 9\,H^2)^2.$$

The quantity D plays a crucial role in the analysis, and there are other ways to compute it. (For sound technical reasons, the sign of D has been switched from the definitions in a couple published papers.)

2. Still using the above notation, define the following quantities for the **LO** and **RO** problems:

$$j_2 = (1 - C_2)\,(1 - C_3)\; x_2\,,$$

$$k_2 = (1 - C_2)\,(1 - C_3)\; y_2\,,$$

$$m_2 = (1 - C_3)\,(C_2 - \cos 2\angle B)\,,$$

$$j_3 = (1 - C_2)\,(1 - C_3)\; x_3\,,$$

$$k_3 = (1 - C_2)\,(1 - C_3)\; y_3\,,$$

$$m_3 = (1 - C_2)\,(C_3 - \cos 2\angle C)\,,$$

$$q = j_2^2\, m_3^2 + k_2^2\, m_3^2 + j_3^2\, m_2^2 + k_3^2\, m_2^2 - j_2^2\, k_3^2 - j_3^2\, k_2^2 + 2\, j_2\, k_2\, j_3\, k_3 - 2\, j_2\, m_2\, j_3\, m_3 - 2\, k_2\, m_2\, k_3\, m_3$$

(*Note:* $\cos 2\angle B = \cos(\phi_1 - \phi_3)$ and $\cos 2\angle C = \cos(\phi_1 - \phi_2)$.)

3. Define these additional quantities for the **RO** problem:

$$\hat{c}_2 = \operatorname{sign}(c_1)\,\Big\{\,\Big(\,[\,\cos(\phi_1 - \phi_2) - \cos(\phi_1 - \phi_3)\,]\,C_1 + \cos(\phi_1 - \phi_3) - \cos(\phi_1 - \phi_2)\cos(\phi_2 - \phi_3) + |\sin((\phi_2 - \phi_3)/2)|\,\sin(\phi_1 - \phi_2)\,[\,2\,(\,1 + \cos(\phi_2 - \phi_3) - 2\,C_1\,)\,]^{1/2}\,\Big) \Big/ \Big(\,1 - C_1 + \cos(\phi_1 - \phi_2)\,C_1 - \cos(\phi_1 - \phi_2)\cos(\phi_2 - \phi_3) + |\sin((\phi_2 - \phi_3)/2)|\,\sin(\phi_1 - \phi_2)\,[\,2\,(\,1 + \cos(\phi_2 - \phi_3) - 2\,C_1\,)\,]^{1/2}\,\Big)\,\Big\}^{1/2},$$

$$\hat{c}_3 = \{ \; (\; [\cos(\phi_1-\phi_3) - \cos(\phi_1-\phi_2)\,]\,C_1 + \cos(\phi_1-\phi_2) - \cos(\phi_1-\phi_3)\cos(\phi_2-\phi_3)$$
$$+ \; |\sin((\phi_2-\phi_3)/2)| \; \sin(\phi_1-\phi_3) \; [\,2\,(\,1+\cos(\phi_2-\phi_3)-2\,C_1\,)\,]^{1/2}\;)$$
$$/ \;\; (\,1-C_1+\cos(\phi_1-\phi_3)\,C_1-\cos(\phi_1-\phi_3)\cos(\phi_2-\phi_3)$$
$$+ \; |\sin((\phi_2-\phi_3)/2)| \; \sin(\phi_1-\phi_3) \; [\,2\,(\,1+\cos(\phi_2-\phi_3)-2\,C_1\,)\,]^{1/2}\;)\;\}^{1/2} \;,$$
$$\check{c}_2 = \{ \; (\; [\cos(\phi_1-\phi_2) - \cos(\phi_1-\phi_3)\,]\,C_1 + \cos(\phi_1-\phi_3) - \cos(\phi_1-\phi_2)\cos(\phi_2-\phi_3)$$
$$- \; |\sin((\phi_2-\phi_3)/2)| \; \sin(\phi_1-\phi_2) \, [\,2\,(\,1+\cos(\phi_2-\phi_3)-2\,C_1\,)\,]^{1/2}\;)$$
$$/ \;\; (\,1-C_1+\cos(\phi_1-\phi_2)\,C_1-\cos(\phi_1-\phi_2)\cos(\phi_2-\phi_3)$$
$$- \; |\sin((\phi_2-\phi_3)/2)| \; \sin(\phi_1-\phi_2) \, [\,2\,(\,1+\cos(\phi_2-\phi_3)-2\,C_1\,)\,]^{1/2}\;)\;\}^{1/2} \;,$$
$$\check{c}_3 = \operatorname{sign}(c_1)\,\{ \; (\; [\cos(\phi_1-\phi_3) - \cos(\phi_1-\phi_2)\,]\,C_1 + \cos(\phi_1-\phi_2) - \cos(\phi_1-\phi_3)\cos(\phi_2-\phi_3)$$
$$- \; |\sin((\phi_2-\phi_3)/2)| \; \sin(\phi_1-\phi_3) \; [\,2\,(\,1+\cos(\phi_2-\phi_3)-2\,C_1\,)\,]^{1/2}\;)$$
$$/ \;\; (\,1-C_1+\cos(\phi_1-\phi_3)\,C_1-\cos(\phi_1-\phi_3)\cos(\phi_2-\phi_3)$$
$$- \; |\sin((\phi_2-\phi_3)/2)| \; \sin(\phi_1-\phi_3) \; [\,2\,(\,1+\cos(\phi_2-\phi_3)-2\,C_1\,)\,]^{1/2}\;)\;\}^{1/2} \;.$$

(When these quantities are required, it will always be possible to select a non-negative real radical whenever a radical is required, and this should always be done.)

Here now are the proposed solutions to each of the proposed problems, excluding the **LA** problem, whose solution is trivial. It is tacitly assumed here that the six quantities $\angle A$, $\angle B$, $\angle C$, α , β and γ satisfy the properties indicated in the first paragraph of this document. The four inequalities concerning α , β and γ are together equivalent to the single condition that $H > 0$, which is straightforward to check.

Proposed solution system for RA:

The question has an affirmative answer if and only if

$$[\;\alpha < \angle A \;\rightarrow\; \cos\angle C \;\, c_2 \;+\; \cos\angle B \;\, c_3 \;>\; 0\;] \;\;\wedge$$
$$[\;\beta < \angle B \;\rightarrow\; \cos\angle A \;\, c_3 \;+\; \cos\angle C \;\, c_1 \;>\; 0\;] \;\;\wedge$$
$$[\;\gamma < \angle C \;\rightarrow\; \cos\angle B \;\, c_1 \;+\; \cos\angle A \;\, c_2 \;>\; 0\;] \;\;\wedge$$
$$[\,(\,\alpha < \angle A \;\wedge\; \beta > \angle B \;\wedge\; \gamma > \angle C\,) \;\rightarrow\; D > 0\,] \;\;\wedge$$
$$[\,(\,\alpha > \angle A \;\wedge\; \beta < \angle B \;\wedge\; \gamma > \angle C\,) \;\rightarrow\; D > 0\,] \;\;\wedge$$
$$[\,(\,\alpha > \angle A \;\wedge\; \beta > \angle B \;\wedge\; \gamma < \angle C\,) \;\rightarrow\; D > 0\,] \;.$$

Note: the arrows mean logical implication and, of course, $p \rightarrow q$ is logically equivalent to $\neg\, p \vee q$.

Proposed solution system for LO:

The question has a negative answer if and only if

$D > 0 \ \wedge \ [\ q < 0 \ \vee \ (2 \, C_2 < 1 + \cos 2\angle B \ \wedge \ (1 + \cos 2\angle A) \ C_3 < \cos 2\angle A + \cos 2\angle C) \ \vee$

$(2 \, C_3 < 1 + \cos 2\angle C \ \wedge \ (1 + \cos 2\angle A) \ C_2 < \cos 2\angle A + \cos 2\angle B) \ ] .$

(Everything here except the "$D > 0$" part describes a region in the $c_2 c_3$ – plane, the projection onto this plane of the non-feasible points. Singular behavior, due to jump discontinuities, is involved here.)

Proposed solution system for RO:

The question has a negative answer if and only if the **LO** question has a negative answer or one of the following hold:

(*i*) $c_1 < \cos \angle A \ \wedge \ [\ c_2 > \cos \angle B \ \vee c_3 > \cos \angle C \ ] ;$

(*ii*) $c_1 > -\cos \angle A \ \wedge [\ (c_2 < 0 \ \wedge \ c_3 < \cos \angle C) \ \vee \ (c_3 < 0 \ \wedge \ c_2 < \cos \angle B)$

$\vee \ (c_2 < \cos \angle B \ \wedge \ c_3 < \cos \angle C \ \wedge \ D < 0) \ ] ;$

(*iii*) $\cos \angle A < c_1 < -\cos \angle A \ \wedge \ [\ (\cos \angle C \, c_2 + \cos \angle B \, c_3 < 0 \ ) \ \vee \ (\ c_2 < \hat{c}_2 \ \wedge \ c_3 < \hat{c}_3 \ )$

$\vee \ (\ c_2 < \check{c}_2 \ \wedge \ c_3 < \check{c}_3 \ ) \ \vee \ (\ c_2 < \hat{c}_2 \ \wedge \ c_3 < \cos \angle C \ \wedge \ D < 0 \ )$

$\vee \ (\ c_3 < \check{c}_3 \ \wedge \ c_2 < \cos \angle B \ \wedge \ D < 0 \ ) \ ] .$

The next few pages contain some figures that show typical experimental results, using Mathematica. Each figure deals with one of the non-trivial problems, using some chosen triangle angles $\angle A$, $\angle B$ and $\angle C$, together with a chosen value of γ (and corresponding value of c_3). Two regions are plotted in the (c_1, c_2) – plane. The first of these regions (blue) is produced by generating pairs/points (c_1, c_2) that correspond to the values of α and β that occur when a suitable triangle ABC is fit into three lines (or rays) using the given γ. Two of these lines (or rays) stay fixed, while the other one varies. An extension of the method of John Sullivan, as discussed by John Wetzel*, is used here to generate these points. The second of the two regions (yellow) is where the proposed solution system for the problem yields a negative answer. Together, these two regions always appear to partition the interior of the ellipse given by $H > 0$† (for fixed c_3), except possibly for a set of measure zero. Many such plots have been generated, lending support for the claim that the proposed solutions to **LO**, **RA** and **RO** are correct.

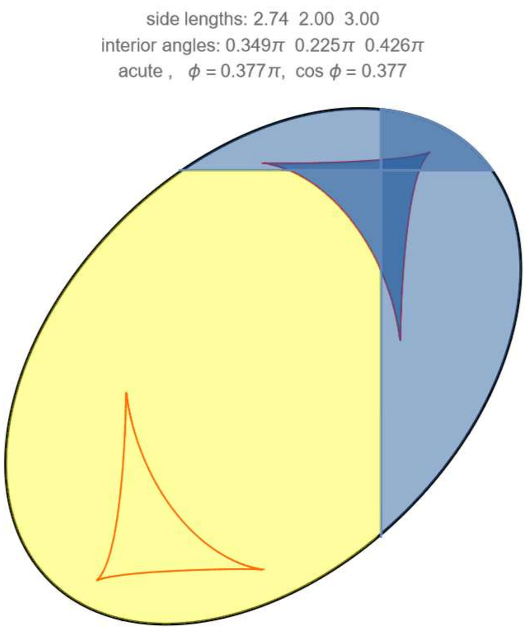
After laboring for quite a long time on these three problems, I will be very surprised if any fundamental flaw is discovered in the proposed solutions, or if any significantly simpler solution is discovered that does not involve solving polynomial equations of degree greater than two (as in the P3P problem). Notice that the proposed solutions to **LO** and **RA** can be expressed as logical combinations of polynomial inequalities in c_1 , c_2 and c_3, without explicit mention of α, β and γ There is no need to extract roots in these systems. On the other hand, the proposed solution to **RO** does require computing square roots. However, it might be possible to find a different solution that avoids this.

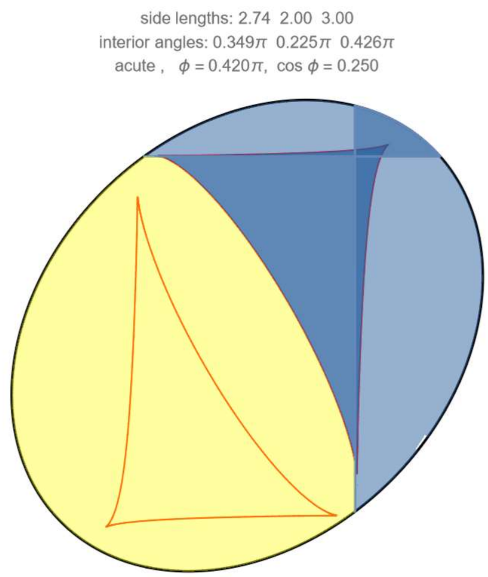
* "An Ancient Elliptic Locus"

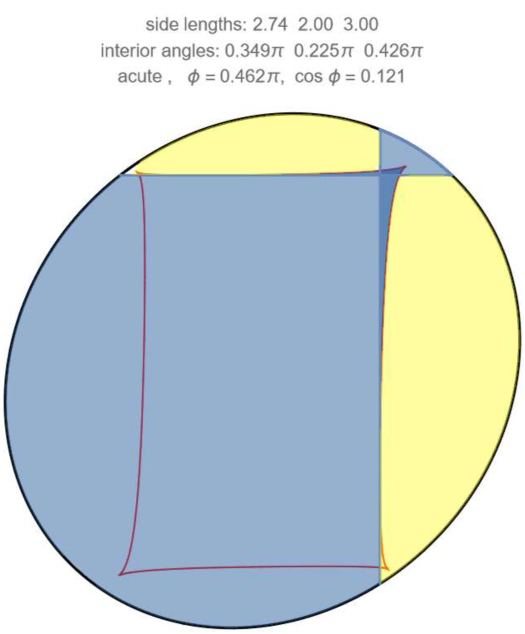
† which is equivalent to $(\alpha + \beta + \gamma < 2\pi) \wedge (\alpha < \beta + \gamma) \wedge (\beta < \gamma + \alpha) \wedge (\gamma < \alpha + \beta)$

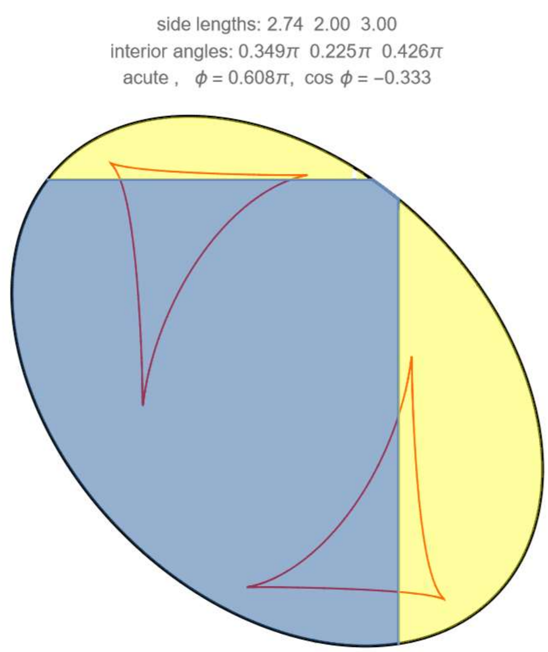
Note: γ is called ϕ in the figures (due to my notation for extending Sullivan's method). Also, the different shades of blue are unimportant for our purposes here. The red curves are the $D = 0$ curves.

<u>Examples of **RA**</u>



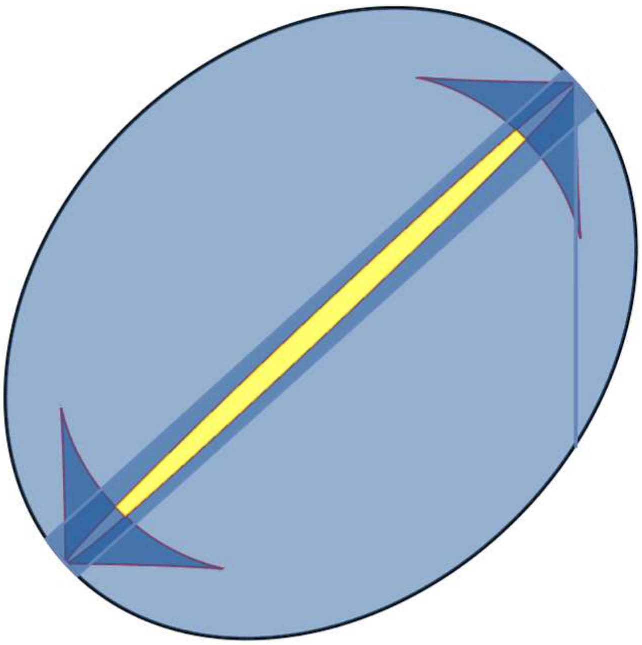




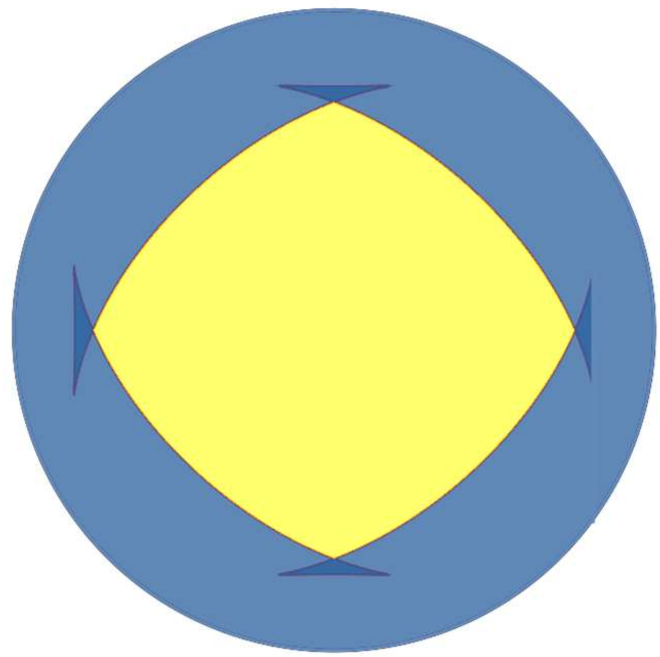


<u>Examples of **LO**</u>

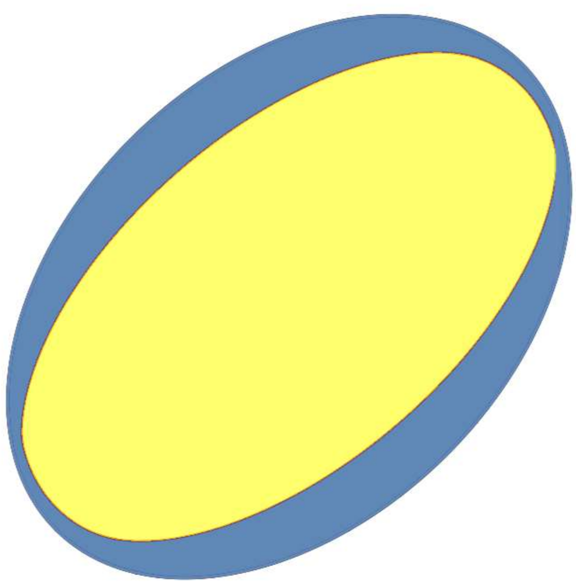
side lengths: 1.81 2.00 3.00
interior angles: 0.200π 0.224π 0.576π
obtuseC , $\phi = 0.425\pi$, $\cos\phi = 0.235$



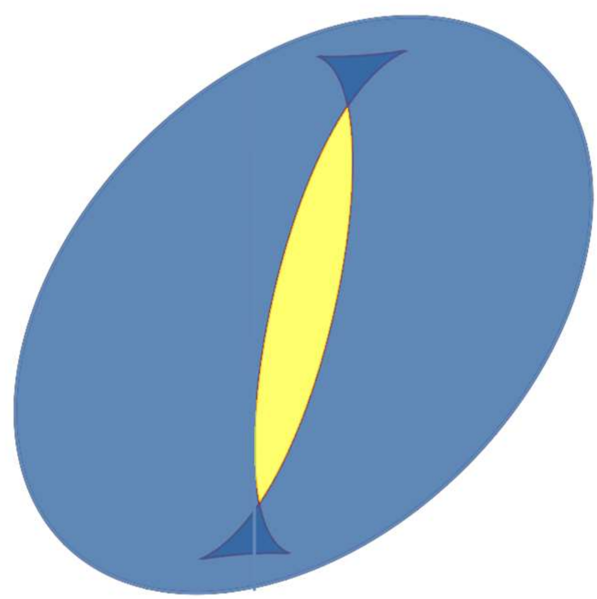
side lengths: 1.81 2.00 3.00
interior angles: 0.200π 0.224π 0.576π
obtuseC , $\phi = 0.500\pi$, $\cos\phi = 0$



side lengths: 1.27 2.00 3.00
interior angles: 0.101π 0.165π 0.734π
obtuseC , $\phi = 0.379\pi$, $\cos\phi = 0.372$

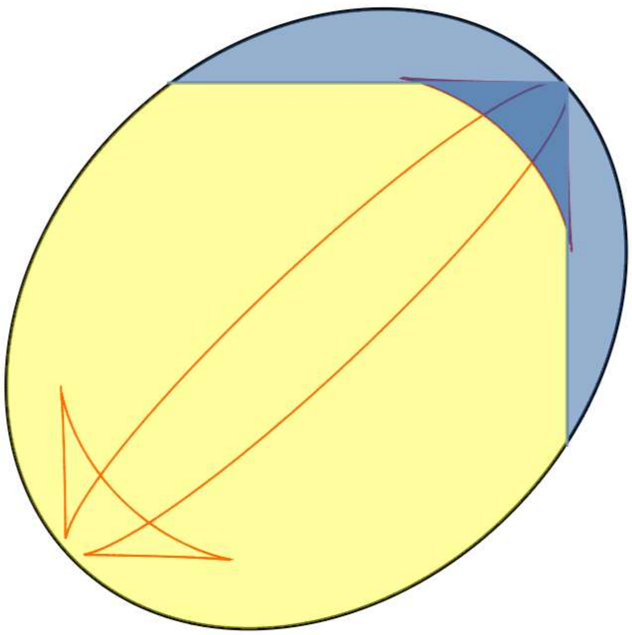


side lengths: 3.88 2.00 3.00
interior angles: 0.554π 0.170π 0.276π
obtuseA , $\phi = 0.379\pi$, $\cos\phi = 0.372$

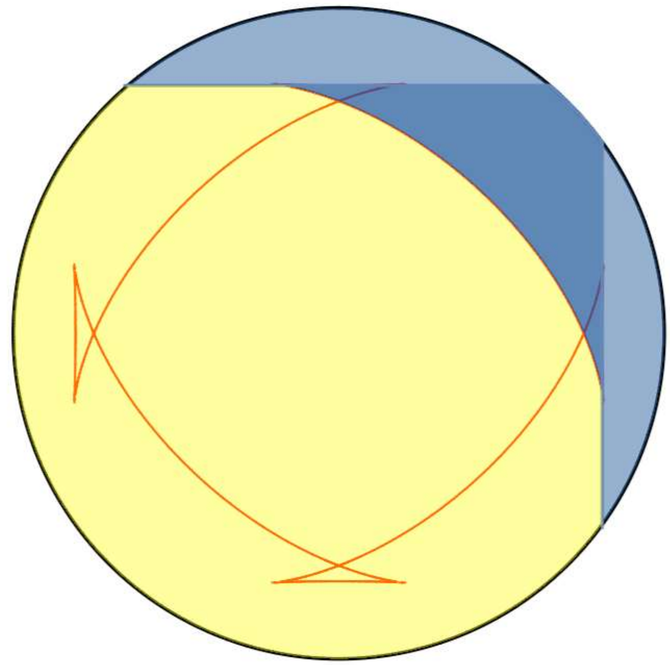


Examples of **RO**

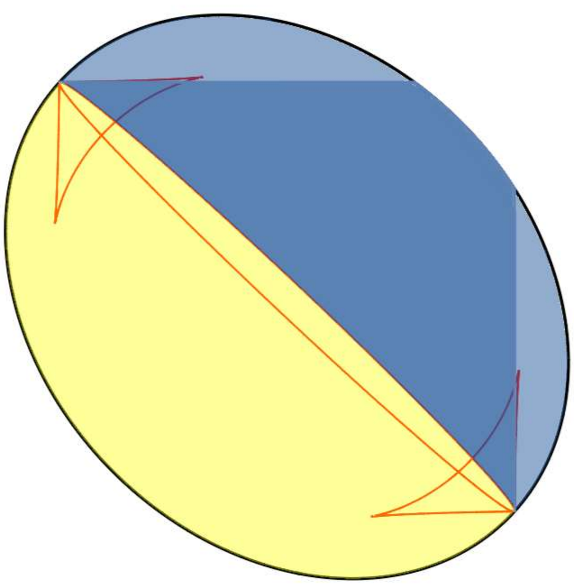
side lengths: 1.81 2.00 3.00
interior angles: 0.200π 0.224π 0.576π
obtuseC , $\phi = 0.430\pi$, $\cos\phi = 0.217$



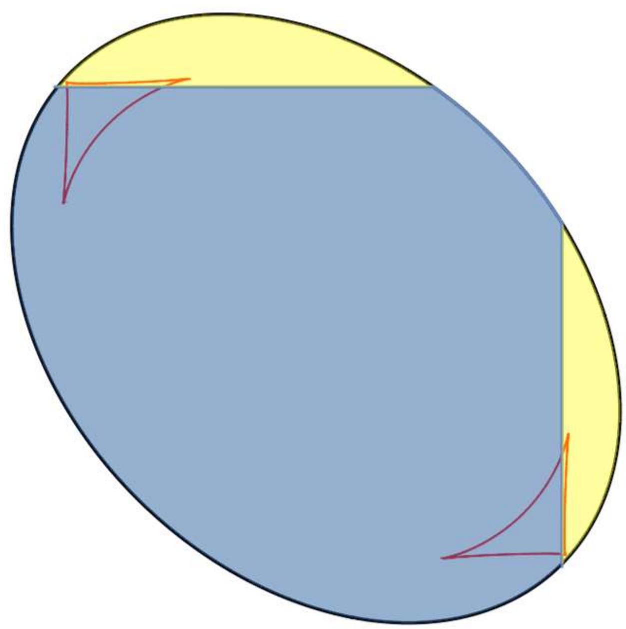
side lengths: 1.81 2.00 3.00
interior angles: 0.200π 0.224π 0.576π
obtuseC , $\phi = 0.500\pi$, $\cos\phi = 0$



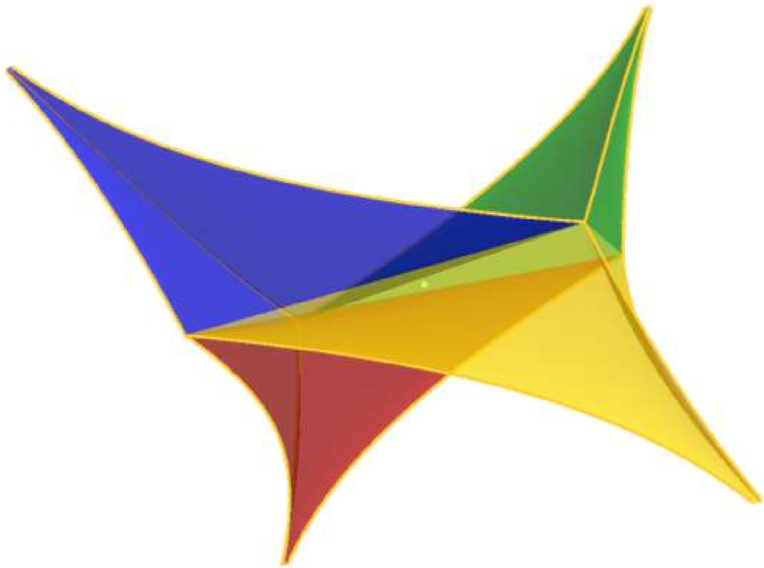
side lengths: 1.81 2.00 3.00
interior angles: 0.200π 0.224π 0.576π
obtuseC , $\phi = 0.575\pi$, $\cos\phi = -0.233$



side lengths: 1.81 2.00 3.00
interior angles: 0.200π 0.224π 0.576π
obtuseC , $\phi = 0.598\pi$, $\cos\phi = -0.304$



Lastly, the surface where $D = 0$ in three-dimensional (c_1, c_2, c_3) –space, when restricted to the region where $|c_j| < 1$ ($j = 1, 2, 3$) and $H > 0$, has a rather interesting shape. It is invariant under the action of the Klein 4-group where this acts by negating any two of the three c's. The following image is typical of the case where all three of the parameter angles $\angle A$, $\angle B$, $\angle C$ are acute:



When one of the angles $\angle A$, $\angle B$, $\angle C$ is obtuse, the surface is self-intersecting, as seen in the following image:

Note: Further figures as well as animations and program code can be obtained at
`https://github.com/mqrieck/tetrahedron_test.cpp`