Angular Properties of a Tetrahedron with an Acute Triangular Base

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Abstract. From a fixed acute triangular base ΔABC, all possible tetrahedra in three-dimensional real space are considered. The possible angles at the additional vertex P are shown to be bounded by certain inequalities, mostly linear inequalities. Together, these inequalities provide fairly tight bounds on the possible angle combinations at P.

Four sets of inequalities are used for this purpose, though the inequalities in the first set are rather trivial. The inequalities in the second set can be established quickly, but do not seem to be known. The third and fourth set of inequalities are proved by studying scalar and vector fields on toroids. The first three sets of inequalities are linear in the angles at P, but the last set involves cosines of these angles. A generalization of the last two sets of inequalities is also proved, using the Poincaré-Hopf Theorem.

Extensive testing of these results has been done using Mathematica and C++. The C++ code for this is listed in an appendix. While it has been demonstrated that the inequalities bound the possible combinations of angles at P, the results also reveal that additional inequalities, in particular linear inequalities, exist that would provided tighter bounds.

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1. Introduction and the Main Result

Using a fixed acute triangle $\triangle ABC$ in three-dimensional real space, consider any point P that is not coplanar with A, B and C. Together, A, B, C and P form the vertices of a tetrahedron. Following standard practice, $\angle A$, $\angle B$ and $\angle C$ will be used to denote the interior angles of $\triangle ABC$. The angles $\angle BPC$, $\angle CPA$ and $\angle APB$ will be denoted α, β and γ, respectively. We will be interested in the limitations that the angles $\angle A$, $\angle B$ and $\angle C$ place on the angles α, β and γ. This turns out to be more complicated than one might initially imagine. All angles will be measured in radians. The side lengths of $\triangle ABC$ opposite A, B and C will be denoted a, b and c, respectively.

Theorem 1.1. *Using the above setup, the following constraints on the possible values of* (α, β, γ) *hold:*

1. $\alpha < \beta + \gamma$, $\beta < \gamma + \alpha$, $\gamma < \alpha + \beta$, $\alpha + \beta + \gamma < 2\pi$,
2. $\angle A + \beta + \gamma < 2\pi$, $\alpha + \angle B + \gamma < 2\pi$, $\alpha + \beta + \angle C < 2\pi$,
3. $\alpha \leqslant \angle A \;\rightarrow\; \beta \leqslant \max\{\angle B, \angle C + \alpha\}$, *and suitable permutations of this,*
4. $\alpha \leqslant \angle A \;\rightarrow\; \cos\angle C \cos\beta + \cos\angle B \cos\gamma > 0$, *and suitable permutations of this.*

"Suitable permutations" means permuting the symbols "A", "B" and "C", and permuting the symbols "α", "β" and "γ", together, in the same way. The right arrows are logical implications. Of course, an implication $p \rightarrow q$ is logically equivalent to the disjunction $\neg p \vee q$. ("$\wedge$", "$\vee$" and "$\neg$" are used here for the basic logical operations of conjunction, disjunction and negation.)

While tetrahedra have been studied in some depth, over many years, the last three sets of constraints in the theorem do not seem to be known, although the constraints in the first set are trivial. For instance, [1] has two substantial chapters devoted to tetrahedra, including various interesting results that generalize classical results on triangles, but does not mention anything like the above constraints on the interior angles of the tetrahedron's faces. These constraints could potentially be of wide practical use in areas such as molecular geometry, tetrahedral finite-element meshes, and camera-tracking, particularly the so-called "perspective 3-point problem" (see [3]).

A Cartesian coordinate system (x, y, z) will be fixed for three-dimensional real space. For simplicity, and without loss of generality, we will henceforth assume that A, B and C are located on the xy-plane, and usually that P lies above this (*i.e.* $z > 0$ for P). However, as part of the analysis, sometimes it will be useful to relax this restriction, and allow P to be on or below the xy-plane. The first two sets of constraints in the theorem are easily established by means of a "spherical model," as follows.

Proof of Parts 1 and 2 of Theorem 1.1. Assume that P is above the xy-plane. Consider a sufficiently small sphere centered at P, and contained in the upper half-space ($z > 0$). The rays $\overrightarrow{PA}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$ intersect the sphere at points A', B' and C' respectively. After rescaling space (for the purpose of this proof only), so that the sphere now has radius one, the great-circle distance between B' and C' is α, and similarly for the other two such pairs of points. Thus, A', B' and C' are the vertices of a spherical triangle whose side lengths are α, β and γ. The first set of constraints (Part 1) in the theorem are simply the well-known constraints for the side lengths of a spherical triangle on a unit sphere.

Next, consider the great circle that is the intersection the sphere and a plane parallel to the xy-plane. The points on this great circle have the same value of z as P. The sideline $\overleftrightarrow{BC}$ of $\triangle ABC$ is parallel to a line through P that cuts this great circle in two antipodal points D and D'. Choose these so that the rays $\overrightarrow{PD}$ and $\overrightarrow{BC}$ point in the same direction. Similarly obtain antipodal points E and E', and antipodal points F and F', with $\overrightarrow{PE}$ and $\overrightarrow{CA}$ pointing in the same direction, and with $\overrightarrow{PF}$ and $\overrightarrow{AB}$ pointing in the same direction.

The plane containing P, B and C also contains B', C', D and D', and we see that these latter four points lie on a common great circle. Likewise for the points C', A', E and E', and for the points A', B', F and F'. Of course the points D, D', E, E', F and F' all lie on the great circle that is parallel to the xy-plane. It is then straightforward to check that great-circle distances between D and E', between E' and F, between F and D', between D' and E, between E and F', and between F' and D are respectively, $\angle C$, $\angle A$, $\angle B$, $\angle C$, $\angle A$ and $\angle B$.

The points A', B' and C' are all located in the lower hemisphere, that is, they have z coordinates that are less that the z coordinate of P. Now, the points A', E' and F form a spherical triangle, and the arc from A' to E' extends the arc from A' to B', while the arc from A' to F extends the arc from A' to C'. Since the distance from A' to B' is γ, and the distance from A' to C' is β, it follows that $\angle A + \beta + \gamma < 2\pi$. Similarly for the other two constraints in Part 2 of the theorem.

□

In order to prove the other two parts of the theorem, it will be helpful to study certain functions (scalar fields) on certain toroids, and their surface gradient vector fields. This forms the content of Sections 2, 3 and 4 of this article. In Section 5, the proof of the remaining parts of Theorem 1.1 will be presented. By a *toroid*, we here mean the surface generated in three-dimesional real space by rotating a circular arc about its endpoints. We will only be interested in toroids whose endpoints are two of the three triangle vertices A, B and C. In fact, we will primarily focus on an arbitrary toroid $\mathcal{T}$ generated by an arc whose endpoints are B and C, and can then argue by symmetry that certain claims about it can be adapted to toroids generated by arcs whose endpoints are A and B, or A and C. We will call B and C the *apexes* of the toroid $\mathcal{T}$.

Lemma 1.2. *Given a toroid $\mathcal{T}$ with apexes B and C, all of the points on $\mathcal{T}$ other than B and C have the same value for the angle $\alpha = \angle BPC$. Moreover, no other point in space has this value for α.*

Proof. By the Inscribed Angle Theorem, all of the points on the arc used to generate $\mathcal{T}$ (with endpoints B and C) have the same value for α. When a point on this arc is rotated about the line $\overleftrightarrow{BC}$, its value of α is clearly unchanged. Thus, the points on $\mathcal{T}$ have the same value for the angle α. Any

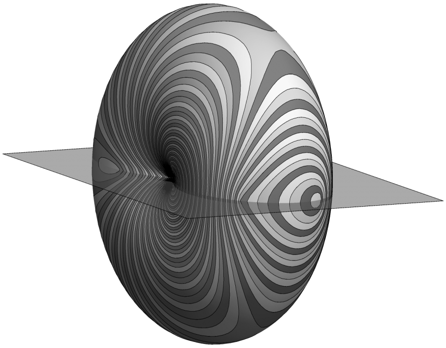

FIGURE 1. Example of $\mathcal{T}or_\alpha$ with contours for a scalar field Q

other point in space, not on $\overleftrightarrow{BC}$, can be rotated about $\overleftrightarrow{BC}$ to a point in the xy-plane that is on the same side of $\overleftrightarrow{BC}$ as the arc used to generate $\mathcal{T}$, and in this half-plane, it will either be inside or outside the arc. If it is inside the arc, then its value of α will be greater than that of the points on the arc. If instead it is outside the arc, then its value of α will be less than that of the points on the arc.

□

Now, given a value for α with $0 < \alpha < \pi$, we will hencefoth let $\mathcal{T}or_\alpha$ denote the toroid consisting of points that have the specified value of α, together with the apexes B and C. We will also fix non-negative numbers μ and ν, not both zero, and will let

$$Q \;=\; \mu\cos\beta + \nu\cos\gamma.$$

We will be interested in studying Q as a scalar field on $\mathcal{T}or_\alpha$ (varying P). Contours for this can be seen in Figure 1. We will also examine the surface gradient vector field $\vec{\triangledown}_\alpha Q$, defined as follows:

$$\vec{\triangledown}_\alpha Q = \vec{\triangledown} Q - \frac{\vec{\triangledown} Q \cdot \vec{\triangledown}\cos\alpha}{\vec{\triangledown}\cos\alpha \cdot \vec{\triangledown}\cos\alpha}\,\vec{\triangledown}\cos\alpha = \frac{-\vec{\triangledown}\cos\alpha \times (\vec{\triangledown}\cos\alpha \times \vec{\triangledown} Q)}{\vec{\triangledown}\cos\alpha \cdot \vec{\triangledown}\cos\alpha}, \tag{1.1}$$

where the other gradients involved in this are gradients in three-dimensional space. Of course, $\vec{\triangledown}_\alpha Q$ at a point on $\mathcal{T}or_\alpha$ will be tangent to this surface. Similarly define the surface gradient $\vec{\triangledown}_\alpha s$ for any scalar field s in three-dimensional space. Note that in the formula for $\vec{\triangledown}_\alpha Q$, we could substitute $\vec{\triangledown}\alpha$ in place of $\vec{\triangledown}\cos\alpha$. However, in the analysis presented in the next few sections, it is more convenient to work with $\vec{\nabla}\cos\alpha$, $\vec{\nabla}\cos\beta$ and $\vec{\nabla}\cos\gamma$, rather than $\vec{\nabla}\alpha$, $\vec{\nabla}\beta$ and $\vec{\nabla}\gamma$.

We will now begin a careful analysis of the vector field $\vec{\nabla}_\alpha Q$ on $\mathcal{T}or_\alpha$, and its implication for the scalar field Q. However, this will be limited to the case where $\alpha < \angle A$, and where the triangle $\triangle ABC$ is acute. In Section 2, the apexes B and C of $\mathcal{T}or_\alpha$ are investigated. Since these are the two points where $\mathcal{T}or_\alpha$ is not smooth, some extra care needs to be taken. Upon replacing tiny cone-like neighborhoods of these points with tiny smooth surfaces that avoid B and C, it will be discovered that the winding number of $\vec{\nabla}_\alpha Q$ around the circular boundary of either of these tiny surfaces is always 0, 1 or 2. In this way, each of B and C can be regarded as a singularity for $\vec{\nabla}_\alpha Q$ on $\mathcal{T}or_\alpha$ of index 0, 1 or 2. (Winding numbers and indexes of singularities are discussed in [2].)

In Section 3, it will be shown that apart from B and C, there is always one or two (literally) other singularities for $\vec{\nabla}_\alpha Q$ on $\mathcal{T}or_\alpha$, lying in the xy-plane, and that these are necessarily local maximum points for Q. In Section 4, additional singularities, not in the xy-plane, are considered. It is shown that there is at most one such singularity in the upper half of the toroid ($z > 0$), and a corresponding singularity in the lower half ($z < 0$). Using the Poincaré-Hopf Theorem (see [2]), it is established that these cannot be points where Q takes on a local extreme value. Thus, Q cannot have an extreme value at a point on $\mathcal{T}or_\alpha$ that is not on the xy-plane. In Section 5, this fact is exploited to establish the third and fourth parts of Theorem 1.1.

2. Toroid Apexes as Singularities

In order to make headway in proving the remaining constraints in Theorem 1, it will be helpful to employ some vector analysis. The quantities $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ will be considered as functions of the variable point $P = (x, y, z)$ in three-dimensional real space, and as such, regarded as scalar fields. Their gradients, $\vec{\nabla}\cos\alpha$, $\vec{\nabla}\cos\beta$ and $\vec{\nabla}\cos\gamma$, are particularly important vector fields here. Let $\vec{s_1}$, $\vec{s_2}$ and $\vec{s_3}$ be the vector fields $\overrightarrow{PA}$, $\overrightarrow{PB}$ and $\overrightarrow{PC}$, respectively. Let $s_j = |\vec{s_j}|$ and $\hat{s_1} = \vec{s_j}/s_j$ $(j = 1, 2, 3)$. A series of lemmas will now be stated and proved.

Lemma 2.1. $\vec{\nabla}s_j = -\hat{s_j}$, $\vec{\nabla}\cdot\hat{s_j} = -2/s_j$ *and* $\vec{\nabla}\times\hat{s_j} = 0$ $(j = 1, 2, 3)$. *Also,* $\hat{s_2}\cdot\hat{s_3} = \cos\alpha = (s_2^2+s_3^2-a^2)\,/\,(2s_2s_3)$, $\hat{s_3}\cdot\hat{s_1} = \cos\beta = (s_3^2+s_1^2-b^2)\,/\,(2s_3s_1)$ *and* $\hat{s_3}\cdot\hat{s_1} = \cos\gamma = (s_1^2+s_2^2-c^2)\,/\,(2s_1s_2)$.

Proof. First consider the vector field $\vec{s} = -(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$, the scalar field $s = |\vec{s}| = \sqrt{x^2+y^2+z^2}$, and the unit vector field $\hat{s} = \vec{s}/s$. Straightforward computations show that $\vec{\nabla}s = -\hat{s}$, $\vec{\nabla}\cdot\vec{s} = -3$, $\vec{\nabla}\cdot\hat{s} = (s\,\vec{\nabla}\cdot\vec{s} - \vec{s}\,\cdot$

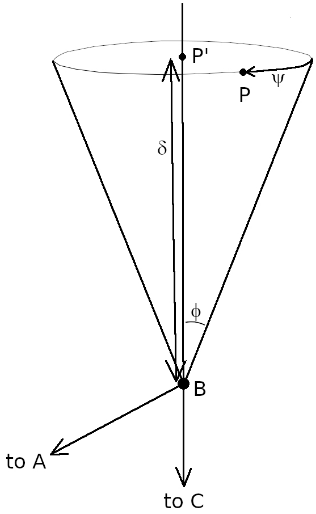

FIGURE 2. A cone near the apex B

$\vec{\bigtriangledown} s)/s^2 = -2/s$, $\vec{\bigtriangledown} \times \vec{s} = 0$, and $\vec{\bigtriangledown} \times \hat{s} = \vec{\bigtriangledown}(s^{-1}) \times \vec{s} + s^{-1} \vec{\bigtriangledown} \times \vec{s} = s^{-2}\, \hat{s} \times \vec{s} = 0$. Now, the vector field $\vec{s_j}$ $(j = 1, 2, 3)$ is simply a translation of the vector field $\vec{s}$, and thus the first few formulas in the lemma immediately follow. The remaining ones hold by a basic property of dot products.

□

Lemma 2.2.

$$\begin{aligned} \vec{\bigtriangledown} \cos\alpha &= \frac{1}{s_2 s_3} \Big[(s_3 \cos\alpha - s_2)\, \hat{s_2} \;+\; (s_2 \cos\alpha - s_3)\, \hat{s_3} \Big] \\ &= \frac{s_3^2 - s_2^2 - a^2}{2 s_2^2 s_3}\, \hat{s_2} \;+\; \frac{s_2^2 - s_3^2 - a^2}{2 s_2 s_3^2}\, \hat{s_3}\,, \end{aligned}$$

and similarly for $\vec{\bigtriangledown} \cos\beta$ *and* $\vec{\bigtriangledown} \cos\gamma$.

Proof. Using calculus and the Law of Cosines, we see that $\vec{\bigtriangledown} \cos\alpha = \vec{\bigtriangledown}\,[\,(s_2^2 + s_3^2 - a^2)/(2s_2 s_3)\,] = [\,(-2s_2\hat{s_2} - 2s_3\hat{s_3})(2s_2 s_3) - (s_2^2 + s_3^2 - a^2)(-2s_3\hat{s_2} - 2s_2\hat{s_3})\,]\,/\,(4s_2^2 s_3^2) =$

$$\frac{-1}{s_2 s_3} \left[\frac{s_2^2 - s_3^2 + a^2}{2s_2}\, \hat{s_2} \;+\; \frac{s_3^2 - s_2^2 + a^2}{2s_3}\, \hat{s_3} \right] =$$

$$\frac{-1}{s_2 s_3}\, [\,(s_2 - s_3 \cos\alpha)\, \hat{s_2} \;+\; (s_3 - s_2 \cos\alpha)\, \hat{s_3}\,],$$

from which, the claims in the lemma immediately follow.

□

In the present section only, we will require a certain cone-based coordinate system, centered at B, and associated basis vectors, as follows. Let's begin with cylindrical coordinates centered at B such that the axis of the cylinders is $\overleftrightarrow{BC}$. Given a point P in space, consider its orthogonal projection P' onto $\overleftrightarrow{BC}$, and let $\rho = |\overline{PP'}|$. Let δ be the signed distance from B to P', signed so that C is in the negative direction. Consider the cylinder of radius ρ with axis $\overleftrightarrow{BC}$. This cylinder intersects the xy-plane in two lines. Consider the one that is on the side of $\overleftrightarrow{BC}$ that is opposite to the side containing A, and call this line ℓ. Let ψ be the signed angle, between $-\pi$ and π, that is made when moving perpendicular to $\overleftrightarrow{BC}$, along the cylinder, from a point on ℓ to P.

The triple (δ, ρ, ψ) are the desired cylindrical coordinates. To convert these to the cone-based coordinates, set $\phi = \arctan(\rho\,/\,|\delta|)$, and use the triple (δ, ϕ, ψ). Our attention will be restricted to points for which $\delta > 0$ and this will tacitly be assumed. We are actually only interested in points P for which δ is a small positive number. See Figure 2.

Next, define some vectors related to the cone-based coordinates, as follows:

$$\mathring{s}_2 = \frac{\hat{s}_2 \times \overrightarrow{BC}}{|\hat{s}_2 \times \overrightarrow{BC}|} \quad , \quad \check{s}_2 = \mathring{s}_2 \times \hat{s}_2 \; .$$

$\overrightarrow{BC} = \vec{s}_3 - \vec{s}_2$ is the vector pointing from B to C, of length a. Of course, the unit vector $\hat{s}_2$ points from P towards B. Consider the cone containing P that has apex B and axis $\overleftrightarrow{BC}$, as in Figure 2. The unit vector $\mathring{s}_2$ points along the circle on the cone that contains P and is orthogonal to $\overleftrightarrow{BC}$, and *w.l.o.g.*, assume this is in the direction of increasing ψ. The unit vector $\check{s}_2$ points orthogonal to the tangent plane for the cone at P, pointing out of the cone. Together, $\{\hat{s}_2, \check{s}_2, \mathring{s}_2\}$ constitute an orthonormal basis for the vector space we get by using P as the origin. The next few lemmas are steps in the goal of expressing $\vec{\nabla}_\alpha \cos\beta$ and $\vec{\nabla}_\alpha \cos\gamma$ in terms of $\delta, \phi, \psi, \hat{s}_2, \check{s}_2$ and $\mathring{s}_2$, for small positive δ (so P near apex B).

Lemma 2.3.

$$\begin{aligned} \vec{s}_1 &= (\delta \sec\phi + c\cos\angle B \cos\phi + c \sin\angle B \sin\phi \cos\psi)\, \hat{s}_2 \\ &+ c\,(\cos\angle B \sin\phi - \sin\angle B \cos\phi \cos\psi)\, \check{s}_2 \\ &- c \sin\angle B \sin\psi\, \mathring{s}_2 \quad \textit{and} \\ \vec{s}_3 &= (\delta\sec\phi + a\cos\phi)\, \hat{s}_2 + a \sin\phi\, \check{s}_2 \end{aligned}$$

Proof. We begin with the vector $\vec{s}_3$ since this is easier to handle. Consider the right triangle having the segment $\overline{BC}$ as hypotenuse, and having one of its legs along the line $\overleftrightarrow{BP}$. This leg has length $a\cos\phi$, and the other leg

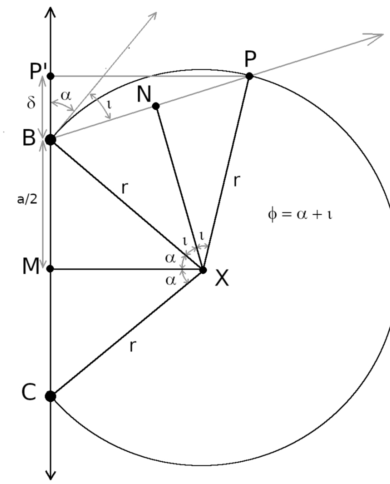

FIGURE 3. Relationship between certain angles

has length $a \sin\phi$. $\hat{s}_2$ is parallel to the former leg, and $\check{s}_2$ is parallel to the latter leg. At P, the unit vector $\check{s}_2$ points out of the constant-(δ, ϕ) cone containing P. It follows that $\overrightarrow{BC} = \vec{s}_3 - \vec{s}_2 = a\cos\phi\,\hat{s}_2 + a\sin\phi\,\check{s}_2$. Observe that $s_2 = \delta\sec\phi$. Thus, $\vec{s}_3 = \vec{s}_2 + \overrightarrow{BC} = \delta\sec\phi\,\hat{s}_2 + a\cos\phi\,\hat{s}_2 + a\sin\phi\,\check{s}_2 = (\delta\sec\phi + a\cos\phi)\,\hat{s}_2 + a\sin\phi\,\check{s}_2$.

Now we will focus on the vector $\vec{s}_1$. Let P' be the orthogonal projection of P onto the line $\overleftrightarrow{BC}$. Observe that $\overrightarrow{PP'} = \rho\,\hat{n} = \delta\tan\phi\,\hat{n}$, where $\hat{n} = \sin\phi\,\hat{s}_2 - \cos\phi\,\check{s}_2$, a unit vector which at P, points towards P'. Also, $\overrightarrow{P'B} = \delta\,\widehat{BC} = \delta\,(\cos\phi\,\hat{s}_2 + \sin\phi\,\check{s}_2)$. Next, we seek a suitable expression for $\overrightarrow{BA}$. For this purpose, notice that $\hat{m}$, defined as $\cos\psi\,\hat{n} + \sin\psi\,\mathring{s}_2 = \cos\psi\sin\phi\,\hat{s}_2 - \cos\psi\cos\phi\,\check{s}_2 + \sin\psi\,\mathring{s}_2$, is a unit vector that is in the xy-plane, perpendicular the vector $\overrightarrow{BC}$, and points in the general direction of A (as opposed to away from it). It can now be checked that $\overrightarrow{BA} = c\,(\cos\angle B\,\widehat{BC} + \sin\angle B\,\hat{m}) = c\,[\cos\angle B\,(\cos\phi\,\hat{s}_2 + \sin\phi\,\check{s}_2) + \sin\angle B\,(\cos\psi\sin\phi\,\hat{s}_2 - \cos\psi\cos\phi\,\check{s}_2 - \sin\psi\,\mathring{s}_2)]$. So now, $\vec{s}_1 = \overrightarrow{PA} = \overrightarrow{PP'} + \overrightarrow{P'B} + \overrightarrow{BA}$, and the claim made in the lemma about this vector follows directly, with a little trigonometry.

□

The constant-ϕ cone containing P, near B, is intended to be an approximation to the portion of the toroid $\mathcal{T}or_\alpha$ containing P. Its value of α is approximately equal to the angle ϕ associated with the cone. The next proposition provides the precise relationship between ϕ and α.

Proposition 2.4. *A point P whose cone-based coordinates (with respect to B, as defined above) are (δ, ϕ, ψ) lies on the toroid $\mathcal{T}or_\alpha$ for the value of α that satisfies*

$$\tan\alpha = \frac{a\cos\phi\sin\phi}{\delta + a\cos^2\phi} \quad \textit{and} \quad \tan(\phi-\alpha) = \frac{\delta\tan\phi}{a+\delta}\ . \tag{2.1}$$

Proof. By rotating about $\overleftrightarrow{BC}$ as needed, we may assume that P is in the xy-plane, and on the circular arc $\mathcal{A}$ that generates $\mathcal{T}or_\alpha$. Let r denote the radius of this circle, and let X denote its center. Let ι be half of the angle $\angle BXP$. See Figure 3.

Since $|\overline{BP}| = \delta\sec\phi$, we see that $2r\,\sin\iota = \delta\sec\phi$. Notice that $r = |\overline{XP}| = |\overline{XB}| = (a/2)\csc\alpha$. Now, $\iota = \phi - \alpha$ by the following reasoning. Let D be an arbitrary point on $\overleftrightarrow{BC}$ but on the side of B that is opposite of the side containing C. Consider also the tangent ray to $\mathcal{A}$ at B, and let E be an arbitrary point along it. This arc is the longer of the two arcs on the circle that connect B and C since $\alpha < \angle A$. Let M be the midpoint of the segment $\overline{BC}$. Note that $\alpha = \angle BPC = \angle MXB = \angle DBE$. But $\phi = \angle DBP$ and $\iota = \angle EBP$, and so $\phi = \alpha + \iota$. This reasoning involves rotating a couple angles by 90 degrees.

We now see that $\delta\sec\phi = 2r\sin\iota = a\csc\alpha\sin(\phi-\alpha) = a\,(\sin\phi\cot\alpha - \cos\phi)$. This can be rewritten as the left equation in (2.1). To show the rest, use $\tan(\phi-\alpha) = (\tan\phi - \tan\alpha)\,/\,(1+\tan\phi\tan\alpha)$, substitute and simplify. $\square$

The following is quickly proved by basic trigonometric reasoning, possibly using the Lemma 2.3.

Lemma 2.5.

$$\begin{cases} s_1^2 &= (c\cos\angle B + \delta)^2 + (c\sin\angle B + \rho\cos\psi)^2 + \rho^2\sin^2\psi \\ &= c^2 + 2c\,\delta\,(\cos\angle B + \sin\angle B\tan\phi\cos\psi) + \delta^2\sec^2\phi \\ s_2^2 &= \delta^2 + \rho^2 \ = \ \delta^2\sec^2\phi \\ s_3^2 &= (a+\delta)^2 + \rho^2 \ = \ a^2 + 2a\delta + \delta^2\sec^2\phi \end{cases}$$

The following two facts about gradients are useful and straightforward to check.

Lemma 2.6.

$$\vec{\nabla}\psi = \frac{1}{\rho}\,\mathring{s}_2 = \frac{\cot\phi}{\delta}\,\mathring{s}_2 \quad \textit{and} \quad \vec{\nabla}\phi = \frac{\cos\phi}{\delta}\,\check{s}_2\ .$$

Now, consider $\vec{\nabla}_\alpha\cos\beta$ and $\vec{\nabla}_\alpha\cos\gamma$ expressed in terms of the basis $\{\hat{s}_2, \mathring{s}_2, \check{s}_2\}$ when δ is a small positive number, and so P is close to B. Substantial manipulations are required to establish the following two lemmas, but this work is straightforward to do, and a sketch for how this might go is provided below. Here R denotes the circumradius of the triangle $\triangle ABC$.

Lemma 2.7. *Write* $\vec{\nabla}_\alpha \cos\beta = \kappa_{\beta 1}\,\hat{s}_2 + \kappa_{\beta 2}\,\check{s}_2 + \kappa_{\beta 3}\,\mathring{s}_2$. *The coefficients here are such that as* $\delta \to 0^+$,

$$\begin{cases} 2R\,\kappa_{\beta 1} &= -\sin^2\angle B \csc\angle C \cos\phi\,(1 + \cot\angle A \tan\phi \cos\psi) \;+\; O(\delta) \\ 2R\,\kappa_{\beta 2} &= -\delta\,\csc\angle A \tan\phi\;\kappa_{\beta 1} \;+\; O(\delta^2) \\ 2R\,\kappa_{\beta 3} &= \cot\angle A \sin^2\angle B \csc\angle C \sin\psi \;+\; O(\delta). \end{cases}$$

Lemma 2.8. *Write* $\vec{\nabla}_\alpha \cos\gamma = \kappa_{\gamma 1}\,\hat{s}_2 + \kappa_{\gamma 2}\,\check{s}_2 + \kappa_{\gamma 3}\,\mathring{s}_2$. *The coefficients here are such that as* $\delta \to 0^+$,

$$\begin{cases} 2R\,\kappa_{\gamma 1} &= -\csc\angle A \sin\angle B \cot\angle C + \cos\angle B\,(\,\cos\angle B \csc\angle C \\ &\quad -\csc\angle A\,)\,\cos^2\phi + \csc\angle A \sin\angle B \csc\angle C \sin(\angle A - \angle B) \\ &\quad \cdot\cos\phi \sin\phi \cos\psi + \sin^2\angle B \csc\angle C \sin^2\phi \cos^2\psi \;+\; O(\delta) \\ 2R\,\kappa_{\gamma 2} &= -\delta\,\csc\angle A \tan\phi\;\kappa_{\gamma 1} \;+\; O(\delta^2) \\ \delta\,\kappa_{\gamma 3} &= \sin\angle B \cos\phi \sin\psi \;+\; O(\delta). \end{cases}$$

Sketch of proof of Lemmas 2.7 and 2.8. Useful substitutions for deriving the claimed formulas come from the previous lemmas and from equation (1.1) with either $\cos\beta$ or $\cos\gamma$ used in place of Q. Each of $\kappa_{\beta 1}$, $\kappa_{\beta 2}$, $\kappa_{\beta 3}$, $\kappa_{\gamma 1}$, $\kappa_{\gamma 2}$ and $\delta\,\kappa_{\gamma 3}$ can be expressed as a Taylor's series in δ. To obtain the claimed formulas, it is only necessary to compute the first nonzero term in each series. □

We are now ready to state and prove the main claim about B and C as singularities for the surface gradient $\vec{\nabla}_\alpha Q$ on the surface $\mathcal{T}or_\alpha$. This will later be needed to prove the last part of Theorem 1.1

Lemma 2.9. *Fix* α *with* $0 < \alpha < \angle A$, *and consider the toroid* $\mathcal{T}or_\alpha$. *Consider a small circle on this toroid near the apex* B. *Specifically, using the cone-based coordinates introduced earlier, the circle should consist of points having the same values of* δ *and* ϕ. *If* δ *is a sufficiently small positive number, then the vector field* $\vec{\nabla}_\alpha Q$ *does not vanish on this circle, and its winding number around this circle, on the surface* $\mathcal{T}or_\alpha$, *is either 0, 1 or 2. Similarly for a small circle near the apex* C.

Proof. It is clear from Lemmas 2.7 and 2.8 that when δ is sufficiently small, and $\sin\psi$ is not comparitively small, the $\mathring{s}_2$ part of $\vec{\nabla}_\alpha Q$ overwhelmingly dominates, provided that $\nu > 0$. This means that the vector $\vec{\nabla}_\alpha Q$ essentially continues to point either forward or backward when moving around the circle, except when $\sin\psi$ becomes small. The only time when $\sin\psi$ becomes sufficiently small so as to affect this behavior is near the xy-plane.

The vector only has a possibility of vanishing at the xy-plane. This would require that $\mu\,\kappa_{\beta 1} + \nu\,\kappa_{\gamma 1} = 0$. If this happens, then simply start over using a smaller value of δ. Since (2.1) indicates how α, δ and ϕ are related, and since we are not changing α, we see that the change in δ will result in a different value for ϕ. It can be checked directly that $\mu\,\kappa_{\beta 1} + \nu\,\kappa_{\gamma 1} = 0$ with $\psi = 0$ or $\psi = \pi$ happens only when $\nu/\mu = -\sin\angle B/\sin(\angle B \mp \phi)$. ($\mu$ and ν are constants here.) Thus, reducing δ to obtain a smaller circle will change ϕ, and thereby eliminating the issue of a vanishing vector.

Now, when approaching the xy-plane, the vector will shrink to a smaller, but nonzero, size, and when crossing the xy-plane, it will make a half rotation before enlarging again and essentially pointing forward or backward again. However, due to the symmetry about the xy-plane, the direction will switch. If the vector was originally pointing forward before getting close to the xy-plane, it will now be pointing backward, and vice-versa. Notice too when ψ is a multiple of π, and so P is on the xy-plane, that while the $\mathring{s}_2$ component of $\vec{\nabla}_\alpha Q$ switches sign due to the $\sin\psi$ factor, the rest of $\vec{\nabla}_\alpha Q$ is nonzero. It is approximately constant in ψ since ψ only occurs in $\cos\psi$ in the part of $\vec{\nabla}_\alpha Q$ that is orthogonal to $\mathring{s}_2$.

It follows that the winding number of $\vec{\nabla}_\alpha Q$ on $\mathcal{T}or_\alpha$ along the constant-(δ, ϕ) circle is either 0, 1 or 2, depending on the directions of the half turns at the two points where the circle intersects the xy-plane, provided that the positive number δ is sufficiently small, and still assuming that $\nu > 0$. If instead, $\nu = 0$ and $\mu > 0$, then an evident adjustment is required.

Finally, by symmetry, this claim can also be made concerning small circles near C.

□

3. Additional Singularities in the xy-plane

As we will see, there is always at least one more singularity of $\vec{\nabla}_\alpha Q$ on $\mathcal{T}or_\alpha$ that lies in the xy-plane, and never more than two. Furthermore, each such singularity corresponds to a local maximum of the function Q on $\mathcal{T}or_\alpha$. To keep the formulas involved in this analysis as simple as possible, it will be assumed in this section (only), without loss of generality, that there is a positive number y_0 such that the coordinates of the points B and C in the xy-plane are $(0, y_0)$ and $(0, -y_0)$.

We continue to assume that the triangle $\triangle ABC$ is acute, and that $0 < \alpha < \angle A$. The intersection of $\mathcal{T}or_\alpha$ and the xy-plane consists of an arc $\mathcal{A}$ connecting B and C on a circle $\mathcal{C}$ whose center X has coordinates $(x_0, 0)$ such that $x_0 > 0$, together with the reflection of this arc about the y-axis. Denote the reflections of $\mathcal{A}$ and $\mathcal{C}$ by $\mathcal{A}'$ and $\mathcal{C}'$, respectively. The radius of $\mathcal{C}$, and of $\mathcal{C}'$ will be denoted by $r_0 = \sqrt{x_0^2 + y_0^2}$. Notice that since $\alpha < \angle A$, the arc $\mathcal{A}$ is the longer of the two arcs on $\mathcal{C}$ joining B and C.

The coordinates of A will just be denoted (x_1, y_1), which are only restricted by above two requirements. The requirement that $\alpha < \angle A$ means that A must be inside the union $\mathcal{A} \cup \mathcal{A}'$. The requirement that $\triangle ABC$ is acute means that $|y_1| < y_0 < r_1$, where $r_1 = \sqrt{x_1^2 + y_1^2}$, as is straighforward to check.

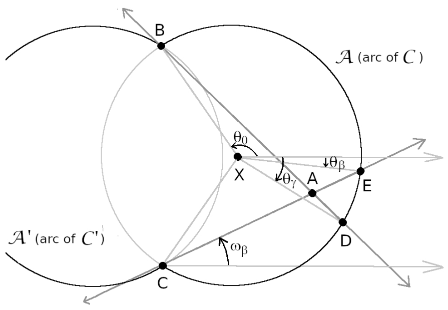

FIGURE 4. The $x_1 > 0$ case of the xy-plane analysis

We will actually limit our focus in the section to singularities of $\vec{\nabla}_\alpha Q$ on the arc $\mathcal{A}$, and basically ignore $\mathcal{A}'$, even though singularities on the latter arc are also significant. By symmetry, studing the singularities on $\mathcal{A}'$ is equivalent to negating x_1, and then focusing on the original arc $\mathcal{A}$. As we will discover, the situation concerning the singularities on $\mathcal{A}$ is quite different depending on whether x_1 is positive or negative. See Figures 4 and 5.

The points on the arc $\mathcal{A}$ will be parameterized via an angle θ with $|\theta| < \theta_0$ with $\theta_0 = \pi - \alpha$. Specifically, the coordinates of such a point P will be $(x_0 + r_0 \cos\theta, r_0 \sin\theta)$. Notice that $(x_0 - r_0 \cos\theta_0, \pm r_0 \sin\theta_0) = (0, \pm y_0)$, the coordinates of B and C. $\mathcal{Tor}_\alpha$ is generated be rotating $\mathcal{A}$ about the y-axis. Letting ψ denote (in this section) the angle of rotation about the y-axis, the points on $\mathcal{Tor}_\alpha$, other than B and C, can be coordinatized via the pair (θ, ψ), with $|\theta| < \theta_0$ and $-\pi < \psi \leqslant \pi$. The next claim is readily checked.

Lemma 3.1. *A point on $\mathcal{Tor}_\alpha$ corresponding to values for θ and ϕ has Cartesian coordinates $(x, y, z) = \big((x_0{+}r_0 \cos\theta) \cos\psi\,,\ r_0 \sin\theta\,,\ (x_0{+}r_0 \cos\theta) \sin\psi \big)$. Moreover,*

$$\begin{cases} s_1^2 &= (x - x_1)^2 + (y - y_1)^2 + z^2 \\ &= r_0^2 + r_1^2 + x_0^2 + 2r_0 x_0 \cos\theta - 2r_0 y_1 \sin\theta - 2x_1(x_0 + r_0\cos\theta)\cos\psi \\ s_{2,3}^2 &= x^2 + (y \mp y_0)^2 + z^2 = 2r_0(r_0 + x_0\cos\theta \mp y_0 \sin\theta) \end{cases}$$

Now recall and apply the Law of Cosine formulas for $\cos\beta$ and $\cos\gamma$ in Lemma 2.1. Using the coordinates system (θ, ψ), observe that $\partial \cos\beta \,/\, \partial\psi$ and $\partial \cos\gamma \,/\, \partial\psi$ are both zero when $\psi = 0$ or $\psi = \pi$ (or any multiple of π), *i.e.* when the point is on the xy-plane. The line $\overleftrightarrow{AB}$ intersects the circle $\mathcal{C}$ in a unique point D (other than the point B), whose coordinates are $(x_0 + r_0 \cos\theta_\gamma, r_0 \sin\theta_\gamma)$ for some θ_γ with $|\theta_\gamma| \leqslant \pi$. Likewise, the line $\overleftrightarrow{AC}$ intersects the circle $\mathcal{C}$ in a unique point$E$ (other than the point C), whose coordinates are $(x_0 + r_0 \cos\theta_\beta, r_0 \sin\theta_\beta)$ for some θ_β with $|\theta_\beta| \leqslant \pi$.

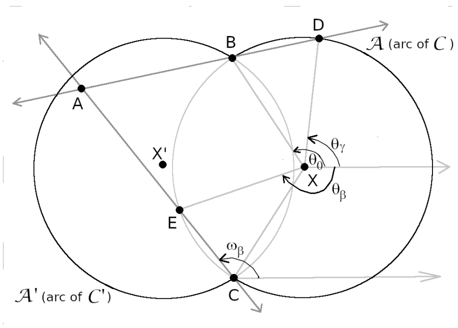

FIGURE 5. The $x_1 < 0$ case of the xy-plane analysis

It is worthwhile to record some equations that corresponds to special possible positions of the point A. Here X denotes the center of the circle $\mathcal{C}$, and X' denotes the center of the circle $\mathcal{C}'$. The claims made here are straightforward to check.

Lemma 3.2.

$$\begin{cases} x_0y_0 - x_0y_1 - y_0x_1 = 0 & \textit{iff } A \textit{ is on the line } \overleftrightarrow{XB} \\ x_0y_0 - x_0y_1 + y_0x_1 = 0 & \textit{iff } A \textit{ is on the line } \overleftrightarrow{X'B} \\ x_0y_0 + x_0y_1 - y_0x_1 = 0 & \textit{iff } A \textit{ is on the line } \overleftrightarrow{XC} \\ x_0y_0 + x_0y_1 + y_0x_1 = 0 & \textit{iff } A \textit{ is on the line } \overleftrightarrow{X'C} \\ y_0^2 + x_0x_1 - y_0y_1 = 0 & \textit{iff } A \textit{ is on the tangent line for } \mathcal{C} \textit{ at } B \\ y_0^2 - x_0x_1 - y_0y_1 = 0 & \textit{iff } A \textit{ is on the tangent line for } \mathcal{C}' \textit{ at } B \\ y_0^2 + x_0x_1 + y_0y_1 = 0 & \textit{iff } A \textit{ is on the tangent line for } \mathcal{C} \textit{ at } C \\ y_0^2 - x_0x_1 + y_0y_1 = 0 & \textit{iff } A \textit{ is on the tangent line for } \mathcal{C}' \textit{ at } C \end{cases}$$

A couple useful facts concerning the signs of various quantities are as follows.

Lemma 3.3.

$$\operatorname{sign}(\theta_0 - |\theta_\beta|) = \operatorname{sign}(x_1)\operatorname{sign}(y_0^2 + x_0x_1 + y_0y_1) \textit{ and}$$
$$\operatorname{sign}(\theta_0 - |\theta_\gamma|) = \operatorname{sign}(x_1)\operatorname{sign}(y_0^2 + x_0x_1 - y_0y_1).$$

Proof. For now, assume that the line $\overleftrightarrow{AC}$ is not tangent to the circle $\mathcal{C}$ at the point C. Whether $\overleftrightarrow{AC}$ intersects $\mathcal{C}$ at a point (other than C) that is on the arc $\mathcal{A}$ or at a point on the other/shorter arc of $\mathcal{C}$ connecting B and C, depends on the sign of x_1 and on whether A is above or below the tangent line to $\mathcal{C}$ at C. This latter condition depends on the sign of $y_0^2 + x_0x_1 + y_0y_1$.

Of course, the sign of $\theta_0 - |\theta_\beta|$ depends on whether the intersection point is on $\mathcal{A}$ or on the other arc. A quick geometric inspection of all four cases makes the first claim clear. The second claim is similarly establed.

□

Definition 3.4. Define the following quantities:

$$\begin{cases} F_\beta &= x_0y_0^2 + x_0y_1^2 - x_0x_1^2 - 2y_0^2x_1 + 2(x_0 - x_1)y_0y_1 \\ G_\beta &= -y_0^3 - y_0y_1^2 + y_0x_1^2 - 2y_0^2y_1 - 2x_0x_1(y_0 + y_1) \\ H_\beta &= r_0\,(y_0^2 + r_1^2 + 2y_0y_1) \\ F_\gamma &= x_0y_0^2 + x_0y_1^2 - x_0x_1^2 - 2y_0^2x_1 + 2(x_1 - x_0)y_0y_1 \\ G_\gamma &= y_0^3 + y_0y_1^2 - y_0x_1^2 - 2y_0^2y_1 + 2x_0x_1(y_0 - y_1) \\ H_\gamma &= r_0\,(y_0^2 + r_1^2 - 2y_0y_1) \end{cases}$$

Definition 3.5. Define the following functions of the angle θ:

$$\begin{cases} p_\beta(\theta) &= x_0y_0 - y_0x_1 + x_0y_1 + r_0(y_0 + y_1)\cos\theta - r_0x_1\sin\theta \\ p_\gamma(\theta) &= -x_0y_0 + y_0x_1 + x_0y_1 + r_0(y_1 - y_0)\cos\theta - r_0x_1\sin\theta \end{cases}$$

The next couple lemmas follow directly from the above definitions and lemmas.

Lemma 3.6.

$$\begin{cases} (x_0\cos\theta_0 + y_0\sin\theta_0 + r_0)(x_0\cos\theta_0 - y_0\sin\theta_0 + r_0) \;=\; (x_0 + r_0\cos\theta)^2 \\ p_\beta(\theta)^2 \;=\; (x_0\cos\theta_0 + y_0\sin\theta_0 + r_0)(F_\beta\cos\theta + G_\beta\cos\theta + H_\beta) \\ p_\gamma(\theta)^2 \;=\; (x_0\cos\theta_0 - y_0\sin\theta_0 + r_0)(F_\gamma\cos\theta + G_\gamma\cos\theta + H_\gamma) \end{cases}$$

Lemma 3.7. *The derivatives of the functions $p_\beta(\theta)$ and $p_\gamma(\theta)$ have the following properties.*

$$\begin{cases} p'_\beta(\theta) \;=\; -r_0(y_0 + y_1)\sin\theta - r_0x_0\cos\theta \\ p'_\beta(-\theta_0) \;=\; y_0^2 + x_0x_1 + y_0y_1 \\ p'_\beta(\theta_\beta) \;=\; -r_0(y_0^2 + x_0x_1 + y_0y_1)(y_0^2 + r_1^2 + 2y_0y_1)\,/\,H_\beta \\ \operatorname{sign}(p'_\beta(-\theta_0)) \;=\; -\operatorname{sign}(p'_\beta(\theta_\beta)) \;=\; \operatorname{sign}(y_0^2 + x_0x_1 + y_0y_1) \\ p'_\gamma(\theta) \;=\; r_0(y_0 - y_1)\sin\theta - r_0x_0\cos\theta \\ p'_\gamma(\theta_0) \;=\; y_0^2 + x_0x_1 - y_0y_1 \\ p'_\gamma(\theta_\gamma) \;=\; -r_0(y_0^2 + x_0x_1 - y_0y_1)(y_0^2 + r_1^2 - 2y_0y_1)\,/\,H_\gamma \\ \operatorname{sign}(p'_\gamma(\theta_0)) \;=\; -\operatorname{sign}(p'_\gamma(\theta_\gamma)) \;=\; \operatorname{sign}(y_0^2 + x_0x_1 - y_0y_1) \end{cases}$$

The sines and cosines of θ_0, θ_β and θ_γ can be expressed as follows.

Lemma 3.8.

$$\begin{cases} \cos\theta_0 \;=\; -x_0/r_0\,,\ \sin\theta_0 \;=\; y_0/r_0 \\ \cos\theta_\beta \;=\; -F_\beta/H_\beta\,,\ \sin\theta_\beta \;=\; -G_\beta/H_\beta \\ \cos\theta_\gamma \;=\; -F_\gamma/H_\gamma \;\; \textit{and} \;\; \sin\theta_\gamma \;=\; -G_\gamma/H_\gamma\,. \end{cases}$$

Proof. The first two equations are immediate. (Note that $\pi/2 < \theta_0 < \pi$.) The equation of $\overleftrightarrow{AB}$ is $(y_1-y_0)x = x_1(y-y_0)$. The equation of $\mathcal{C}$ is $(x-x_0)^2+y^2 = r_0^2$. Eliminating y, and solving for x yields $x = 0$ or $x = 2x_1(x_0x_1 - y_0y_1 + y_0^2)\,/\,(y_0^2 + r_1^2 - 2y_0y_1)$. Using the latter, it can be seen that the desired point (x, y) is such that $(x - x_0, y) = r_0\,(-F_\gamma/H_\gamma, -G_\gamma/H_\gamma)$. The formulas for θ_γ in the lemma now follow. The formulas for θ_β can be similarly proved.

□

Three sinusoidal functions of interest are as follows.

Lemma 3.9.

$$\begin{cases} x_0\cos\theta \pm y_0\sin\theta + r_0 \;=\; 2\,r_0\,\sin^2\frac{\theta\pm\theta_0}{2} \\ F_\beta\cos\theta + G_\beta\sin\theta + H_\beta \;=\; 2H_\beta\sin^2\frac{\theta-\theta_\beta}{2} \\ F_\gamma\cos\theta + G_\gamma\sin\theta + H_\gamma \;=\; 2H_\gamma\sin^2\frac{\theta-\theta_\gamma}{2}\,. \end{cases}$$

Proof. By Lemma 3.8, $(-F_\beta/H_\beta)\cos\theta + (-G_\beta/H_\beta)\sin\theta = \cos\theta_\beta\cos\theta + \sin\theta_\beta\sin\theta = \cos(\theta-\theta_\beta)$. So, $\sin^2[(\theta-\theta_\beta)/2] = [1-\cos(\theta-\theta_\beta)]\,/\,2 = (F_\beta\cos\theta + G_\beta\sin\theta + H_\beta)\,/\,(2H_\beta)$. This proves the equation that involves β. The other equations can be proved in a similar manner.

□

The following formulas for $p_\beta(\theta)$ and $p_\gamma(\theta)$, and their signs, are also useful.

Lemma 3.10.

$$\operatorname{sign}(p_\beta(\theta)) \;=\; \operatorname{sign}(x_1)\operatorname{sign}(\theta_0-\theta_\beta)\operatorname{sign}(\theta_0+\theta)\operatorname{sign}(\theta_\beta-\theta)$$

and

$$\operatorname{sign}(p_\gamma(\theta)) \;=\; \operatorname{sign}(x_1)\operatorname{sign}(\theta_0-\theta_\gamma)\operatorname{sign}(-\theta_0+\theta)\operatorname{sign}(\theta_\gamma-\theta).$$

Proof. From Lemmas 3.6 and 3.9, we see that $p_\beta(\theta) = 0$ if and only if $\theta = -\theta_0$ or $\theta = \theta_\beta$. From Definition 3.5, $p_\beta(\theta)$ is a smooth function, and we see that

$$p_\beta(\theta) \;=\; 2\,\varepsilon_\beta\sqrt{r_0H_\beta}\,\sin\frac{\theta+\theta_0}{2}\,\sin\frac{\theta-\theta_\beta}{2}\ \text{and}$$
$$p_\gamma(\theta) \;=\; 2\,\varepsilon_\gamma\sqrt{r_0H_\gamma}\,\sin\frac{\theta-\theta_0}{2}\,\sin\frac{\theta-\theta_\gamma}{2},$$

where $\varepsilon_\beta, \varepsilon_\gamma \in \{-1, 1\}$ are constants. $p_\beta(\theta)$ changes sign when θ is $-\theta_0$ or θ_β, and nowhere else. $p'_\beta(-\theta_0) = y_0^2 + x_0 x_1 + y_0 y_1$, whose sign is $\operatorname{sign}(x_1)\operatorname{sign}(\theta_0 - |\theta_\beta|)$, by Lemmas 3.3 and 3.7. It follows that $\operatorname{sign}(p_\beta(\theta)) = \operatorname{sign}(x_1)\operatorname{sign}(\theta_0 - |\theta_\beta|) \cdot \operatorname{sign}(\theta_\beta + \theta_0)\operatorname{sign}(\theta_0 + \theta)\operatorname{sign}(\theta_\beta - \theta) = \operatorname{sign}(x_1)\operatorname{sign}(\theta_0 - \theta_\beta)\operatorname{sign}(\theta_0 + \theta_\beta) \cdot \operatorname{sign}(\theta_\beta + \theta_0)\operatorname{sign}(\theta_0 + \theta)\operatorname{sign}(\theta_\beta - \theta) = \operatorname{sign}(x_1)\operatorname{sign}(\theta_0 - \theta_\beta)\operatorname{sign}(\theta_0 + \theta)\operatorname{sign}(\theta_\beta - \theta)$. Similarly for $\operatorname{sign}(p_\gamma(\theta))$.

□

Lemma 3.11.

$$p_\beta(\theta) \;=\; 2\operatorname{sign}(x_1)\operatorname{sign}(\theta_\beta - \theta_0)\sqrt{r_0 H_\beta}\,\sin\frac{\theta + \theta_0}{2}\,\sin\frac{\theta - \theta_\beta}{2}$$

and

$$p_\gamma(\theta) \;=\; 2\operatorname{sign}(x_1)\operatorname{sign}(\theta_\gamma + \theta_0)\sqrt{r_0 H_\gamma}\,\sin\frac{\theta - \theta_0}{2}\,\sin\frac{\theta - \theta_\gamma}{2}\,.$$

Proof. Again, $p_\beta(\theta) = 0$ if and only if $\theta = -\theta_0$ or $\theta = \theta_\beta$. Similarly, $p_\gamma(\theta) = 0$ if and only if $\theta = \theta_0$ or $\theta = \theta_\gamma$. From Definition 3.5, $p_\beta(\theta)$ and $p_\gamma(\theta)$ are smooth functions. From this and from Lemmas 3.6 and 3.9, we see that

$$p_\beta(\theta) \;=\; 2\,\varepsilon_\beta\sqrt{r_0 H_\beta}\,\sin\frac{\theta + \theta_0}{2}\,\sin\frac{\theta - \theta_\beta}{2}\ \text{and}$$

$$p_\gamma(\theta) \;=\; 2\,\varepsilon_\gamma\sqrt{r_0 H_\gamma}\,\sin\frac{\theta - \theta_0}{2}\,\sin\frac{\theta - \theta_\gamma}{2},$$

We also see that $\operatorname{sign}\big(\sin[(\theta + \theta_0)/2]\big) = \operatorname{sign}(\theta + \theta_0)$, and $\operatorname{sign}\big(\sin[(\theta - \theta_\beta)/2]\big) = \operatorname{sign}(\theta - \theta_\beta)$. So, using Lemma 3.10, $\varepsilon_\beta = \operatorname{sign}(x_1)\operatorname{sign}(\theta_0 - \theta_\beta)\operatorname{sign}(\theta_0 + \theta)\operatorname{sign}(\theta_\beta - \theta) \cdot \operatorname{sign}(\theta + \theta_0)\operatorname{sign}(\theta - \theta_\beta) = \operatorname{sign}(x_1)\operatorname{sign}(\theta_\beta - \theta_0)$. Thus we obtain the first equation in the lemma. The second is proved similarly.

□

We will now focus our attention on studying the following functions of θ to gain a better understanding of how Q varies as a function of θ when $\psi = 0$, *i.e.* when the point P is on $\mathcal{C}$.

Definition 3.12.

$$
\begin{cases}
\mathcal{D}_\beta(\theta) & = & \frac{\partial}{\partial\theta}\Big|_{\psi=0}\cos\beta \;=\; \frac{\partial}{\partial\theta}\Big|_{\psi=0}\frac{s_1^2+s_3^2-b^2}{2s_1s_3} \\
\mathcal{D}_\gamma(\theta) & = & \frac{\partial}{\partial\theta}\Big|_{\psi=0}\cos\gamma \;=\; \frac{\partial}{\partial\theta}\Big|_{\psi=0}\frac{s_1^2+s_2^2-c^2}{2s_1s_2} \\
\mathcal{D}(\theta) & = & \frac{\partial}{\partial\theta}\Big|_{\psi=0}Q \;=\; \mu\,\mathcal{D}_\beta(\theta)+\nu\,\mathcal{D}_\gamma(\theta) \\
\mathcal{S}_\beta(\theta) & = & \operatorname{sign}(x_1)\operatorname{sign}(\theta_\beta-\theta_0)\sqrt{y_0^2+r_1^2+2y_0y_1}\,\sin[\tfrac{1}{2}(\theta-\theta_\beta)] \\
\mathcal{S}_\gamma(\theta) & = & -\operatorname{sign}(x_1)\operatorname{sign}(\theta_0+\theta_\gamma)\sqrt{y_0^2+r_1^2-2y_0y_1}\,\sin[\tfrac{1}{2}(\theta-\theta_\gamma)] \\
\mathcal{S}(\theta) & = & \mu\,\mathcal{S}_\beta(\theta)+\nu\,\mathcal{S}_\gamma(\theta)
\end{cases}
$$

Lemma 3.13.

$$\mathcal{D}_\beta(\theta) \;=\; \frac{\operatorname{sign}(p_\beta(\theta))\,(y_0^2-r_1^2+2x_0x_1)\sqrt{F_\beta\cos\theta+G_\beta\sin\theta+H_\beta}}{2\sqrt{2r_0}\,s_1^3},$$

and

$$\mathcal{D}_\gamma(\theta) \;=\; \frac{\operatorname{sign}(p_\beta(\theta))\,(y_0^2-r_1^2+2x_0x_1)\sqrt{F_\gamma\cos\theta+G_\gamma\sin\theta+H_\gamma}}{2\sqrt{2r_0}\,s_1^3}.$$

Proof. Calculations involving substitutions suggested by Lemma 3.1 lead directly to this: $\mathcal{D}_\beta(\theta) \;=\; \{\,[\,-4x_0r_0\sin\theta+2(y_0-y_1)r_0\cos\theta+2x_1r_0\sin\theta\cos\psi\,]\,(2s_1s_3)-(s_1^2+s_3^2-b^2)\,[\,(s_3/s_1)\,(-2x_0r_0\sin\theta-2y_1r_0\cos\theta+2x_1r_0\sin\theta\cos\psi)+(s_1/s_3)(-2x_0r_0\sin\theta+2y_0r_0\cos\theta\,]\,\}\,/\,(4s_1^2s_3^2)$, evaluated at $\psi=0$. Using Definition 3.5 and Lemma 3.6, this expands and then factors to produce $(y_0^2-r_1^2+2x_0x_1)\,[\,(x_0\cos\theta+y_0\sin\theta+r_0)\,p_\beta(\theta)\,]\;/\;[\,2\sqrt{2r_0}\,(x_0\cos\theta+y_0\sin\theta+r_0)^{3/2}\,s_1^3\,]$ $=\operatorname{sign}(p_\beta(\theta))\,(y_0^2-r_1^2+2x_0x_1)\sqrt{F_\beta\cos\theta+G_\beta\cos\theta+H_\beta}\;/\;[\,2\sqrt{2r_0}\,s_1^3\,]$. Similarly for $\mathcal{D}_\gamma(\theta)$. □

The next formulas can be checked directly by applying the half-angle formula for sines.

Lemma 3.14.

$$
\begin{cases}
\sin\frac{\theta_0-\theta_\beta}{2} & = & \frac{\operatorname{sign}(\theta_0-\theta_\beta)\,|x_1|}{\sqrt{r_1^2+y_0^2+2y_0y_1}} \\
\sin\frac{\theta_0+\theta_\gamma}{2} & = & \frac{\operatorname{sign}(\theta_0+\theta_\gamma)\,|x_1|}{\sqrt{r_1^2+y_0^2-2y_0y_1}} \\
\sin\frac{\theta_0+\theta_\beta}{2} & = & \frac{\operatorname{sign}(\theta_0+\theta_\beta)\,|y_0^2+x_0x_1+y_0y_1|}{r_0\sqrt{r_1^2+y_0^2+2y_0y_1}} \\
\sin\frac{\theta_0-\theta_\gamma}{2} & = & \frac{\operatorname{sign}(\theta_0-\theta_\gamma)\,|y_0^2+x_0x_1-y_0y_1|}{r_0\sqrt{r_1^2+y_0^2-2y_0y_1}} \\
\sin\frac{\theta_\beta-\theta_\gamma}{2} & = & \frac{\operatorname{sign}(\theta_\beta-\theta_\gamma)\,y_0\,|r_1^2-y_0^2-2x_0x_1|}{r_0\sqrt{r_1^2+y_0^2+2y_0y_1}\sqrt{r_1^2+y_0^2-2y_0y_1}}.
\end{cases}
$$

We will now discover particularly simple formulas for $\mathcal{D}_\beta(\theta)$, $\mathcal{D}_\gamma(\theta)$ and $\mathcal{D}(\theta)$, provided we restrict θ to have absolute value less than θ_0.

Lemma 3.15. *When $|\theta| < \theta_0$, we have*

$$\begin{cases} \mathcal{D}_\beta(\theta) & = & (y_0^2 - r_1^2 + 2x_0x_1)\ \mathcal{S}_\beta(\theta)\ /\ (2s_1^3) \\ \mathcal{D}_\gamma(\theta) & = & (y_0^2 - r_1^2 + 2x_0x_1)\ \mathcal{S}_\gamma(\theta)\ /\ (2s_1^3) \\ \mathcal{D}(\theta) & = & (y_0^2 - r_1^2 + 2x_0x_1)\ \mathcal{S}(\theta)\ /\ (2s_1^3) \end{cases}$$

Proof. Beginning with Lemma 3.13, and applying Definition 3.4 and Lemmas 3.9 and 3.10, $\mathcal{D}_\beta(\theta) = \text{sign}(p_\beta(\theta))(y_0^2 - r_1^2 + 2x_0x_1)\sqrt{F_\beta \cos\theta + G_\beta \cos\theta + H_\beta}$ $/\ [\,2\sqrt{2r_0}\,s_1^3\,] = \text{sign}(x_1)\,\text{sign}(\theta_0 - \theta_\beta)\,\text{sign}(\theta_0 + \theta)\,\text{sign}(\theta_\beta - \theta)\,\sqrt{2H_\beta}\,(y_0^2 - r_1^2 + 2x_0x_1)\,|\sin[(\theta - \theta_\beta)/2]|\ /\ (2\sqrt{2r_0}\,s_1^3) = \text{sign}(x_1)\,\text{sign}(\theta_\beta - \theta_0)\,\text{sign}(\theta_0 + \theta)\,\sqrt{y_0^2 + r_1^2 + 2y_0y_1}\,(y_0^2 - r_1^2 + 2x_0x_1)\ \sin[(\theta - \theta_\beta)/2]\ /\ (2s_1^3)$. But, $\text{sign}(\theta_0 + \theta) = 1$ since we are assuming that $|\theta| < \theta_0$. This establishes the formula for $\mathcal{D}_\beta(\theta)$. Similarly for $\mathcal{D}_\gamma(\theta)$. From these, the formula for $\mathcal{D}(\theta)$ follows immediately.

□

The next formulas are direct consequences of Definition 3.12 and Lemma 3.14.

Lemma 3.16.

$$\begin{cases} \mathcal{S}_\beta(\theta_0) = -x_1\ ,\ \mathcal{S}_\gamma(\theta_0) \ = \ -\dfrac{y_0^2 + x_0x_1 - y_0y_1}{r_0} \\ \mathcal{S}_\gamma(-\theta_0) = x_1\ ,\ \mathcal{S}_\beta(-\theta_0) = \dfrac{y_0^2 + x_0x_1 + y_0y_1}{r_0} \\ \mathcal{S}_\beta(\theta_\beta) = 0\ ,\ \mathcal{S}_\gamma(\theta_\beta) = \dfrac{\text{sign}\ (\theta_\gamma + \theta_0)\ \text{sign}\ (\theta_\beta - \theta_\gamma)\ y_0(y_0^2 - r_1^2 + 2x_0x_1)}{r_0\sqrt{y_0^2 + r_1^2 + 2y_0y_1}} \\ \mathcal{S}_\gamma(\theta_\gamma) = 0\ ,\ \mathcal{S}_\beta(\theta_\gamma) \ = \ \dfrac{\text{sign}\ (\theta_\beta - \theta_0)\ \text{sign}\ (\theta_\gamma - \theta_\beta)\ y_0(y_0^2 - r_1^2 + 2x_0x_1)}{r_0\sqrt{y_0^2 + r_1^2 - 2y_0y_1}} \end{cases} .$$

With $|\theta| < \theta_0$, we see that $\mathcal{D}(\theta) = 0$ if and only if $\mathcal{S}(\theta) = 0$. Also, $\mathcal{S}(\theta)$ is a sinusoidal function of period 4π that satisfies $\mathcal{S}(\theta + 2\pi) = -\mathcal{S}(\theta)$.

Lemma 3.17. *There is at most one value of θ with $|\theta| < \theta_0$ for which $\mathcal{S}(\theta) = 0$ and hence $\mathcal{D}(\theta) = 0$.*

Proof. In fact, there cannot be more than one value of θ with $|\theta| < \pi$ for which $\mathcal{S}(\theta) = 0$. If there were, then there would also be more than one value of θ with $\pi < \theta < 3\pi$ for which $\mathcal{S}(\theta) = 0$, because $\mathcal{S}(\theta + 2\pi) = -\mathcal{S}(\theta)$. This would mean at least four values of θ with $-\pi < \theta < 3\pi$ for which $\mathcal{S}(\theta) = 0$. This means a sinusoidal function of period 4π (*i.e* a function of the form $\sin[(\theta - \theta_0)/2] + \kappa$) is zero at least four times over a single cycle, which is not possible.

□

Lemma 3.18. $\text{sign}(y_0^2 - r_1^2 + 2x_0x_1) = \text{sign}(x_1)$. *Therefore, if* $|\theta| < \theta_0$, *then* $\text{sign}[\mathcal{D}(\theta)] = \text{sign}(x_1)\,\text{sign}[\mathcal{S}(\theta)]$.

Proof. We continue to assume that $\alpha < \angle A < \pi/2$. Notice that $y_0^2 - r_1^2 + 2x_0x_1 = r_0^2 - (x_1 - x_0)^2 - y_1^2$. The sign of this is +1 if the point A is inside the circle $\mathcal{C}$, but -1 if A is outside $\mathcal{C}$. When $x_1 > 0$, we require that A be inside $\mathcal{C}$ because of the constraint $\alpha < \angle A$. When $x_1 < 0$, it is necessary for A to be outside $\mathcal{C}$ (though it is inside $\mathcal{C}'$) because of the constraint $\angle A < \pi/2$, which means that A must be outside the circle that has the segment $\overline{BC}$ as a diameter. The rest follows from Lemma 3.15.

□

Lemma 3.19. *Assume that* $x_1 < 0$. *Then,*

$$\mathcal{S}(-\theta_0) > 0 \;\Rightarrow\; \mathcal{S}(\theta_0) > 0 \quad \textit{and} \quad \mathcal{D}(-\theta_0) < 0 \;\Rightarrow\; \mathcal{D}(\theta_0) < 0.$$

The equation $\mathcal{S}(\theta) = 0$ *has at most one solution* $\theta = \tilde{\theta}$ *such that* $|\tilde{\theta}| < \theta_0$. *If* $\mathcal{S}(-\theta_0) > 0$ *and* $\mathcal{S}(\theta_0) < 0$, *then there is exactly one such root. If instead,* $\mathcal{S}(-\theta_0) < 0$ *and* $\mathcal{S}(\theta_0) < 0$, *then there are no such roots.*

If $\tilde{\theta}$ *exists and* $|\theta_\beta| < \theta_0$, *then* $\theta_\beta < \tilde{\theta}$. *If* $\tilde{\theta}$ *exists and* $|\theta_\gamma| < \theta_0$, *then* $\tilde{\theta} < \theta_\gamma$. *If* $\tilde{\theta}$ *exists,* $|\theta_\beta| < \theta_0$ *and* $|\theta_\gamma| < \theta_0$, *then* $-\theta_0 < \theta_\beta < \tilde{\theta} < \theta_\gamma < \theta_0$.

Proof. Assume the contrary of the first implication. That is, assume that it is possible to have $\mathcal{S}(-\theta_0) > 0$, but $\mathcal{S}(\theta_0) \leqslant 0$. From Lemma 3.16, $\text{sign}[\,\mathcal{S}(-\theta_0)\,] = \text{sign}[\,(y_0^2 + x_0x_1 + y_0y_1)\,\mu + r_0x_1\,\nu\,]$ and $\text{sign}[\,\mathcal{S}(\theta_0)\,] = -\text{sign}[\,r_0x_1\,\mu + (y_0^2 + x_0x_1 - y_0y_1)\,\nu\,]$. Also, by Lemma 3.18, the signs of $\mathcal{S}(\theta)$ and $\mathcal{D}(\theta)$ are opposite.

Since we are assuming that $\mathcal{S}(-\theta_0) > 0$, we must have $(y_0^2 + x_0x_1 + y_0y_1)\,\mu > -r_0x_1\,\nu$. Since we are assuming that $\mathcal{S}(\theta_0) \leqslant 0$, we must have $(y_0^2 + x_0x_1 - y_0y_1)\,\nu \geqslant -r_0x_1\,\mu$. Now, $(y_0^2 + x_0x_1 + y_0y_1)(y_0^2 + x_0x_1 - y_0y_1) - r_0^2y_1^2 = y_0^2(y_0^2 - r_1^2 + 2x_0x_1 < 0$. This yields a contradiction. The second implication follows from Lemmas 3.18.

By Lemma 3.17, there can be at most one roots between $-\theta_0$ and θ_0. Now, $\mathcal{S}(\theta)$ must have an even number of roots between $-\theta_0$ and θ_0 if $\mathcal{S}(-\theta_0)$ and $\mathcal{S}(\theta_0)$ have the same sign, and hence cannot have any roots at all in this range. However, if the signs are different, then there must have an odd number of roots, and hence a unique root. The remaining claims follow from these facts.

□

Lemma 3.20. *Assume that* $x_1 > 0$. *Then,*

$$\mathcal{S}(-\theta_0) > 0\;,\;\mathcal{S}(\theta_0) < 0\;,\;\mathcal{D}(-\theta_0) > 0 \textit{ and } \mathcal{D}(\theta_0) < 0.$$

The equation $\mathcal{D}(\theta) = 0$ *has exactly one solution* $\theta = \tilde{\theta}$ *such that* $|\tilde{\theta}| < \theta_0$. *Here* $-\theta_0 < \theta_\gamma < \tilde{\theta} < \theta_\beta < \theta_0$.

Proof. Since, $|y_1| < y_0$, we have $y_0^2 + x_0x_1 \pm y_0y_1 > 0$. Thus, $\mathcal{S}(-\theta_0) > 0$ and $\mathcal{S}(\theta_0) < 0$. The rest follows from Lemmas 3.15 through 3.18, using the same reasoning as in the proof of Lemma 3.19.

□

This concludes the analysis of the variation of $Q = \mu\cos\beta + \nu\sin\gamma$ along the arc $\mathcal{A}$, as a function of θ. We next consider how varying ψ at such points, and so moving these points vertically, affects Q. Recall that the first derivatives of Q with respect to ψ vanishes at points on $\mathcal{A}$. We consider the second derivatives now.

Definition 3.21. Define the following:

$$\begin{cases} \mathcal{E}_\beta(\theta) = \frac{\partial^2}{\partial\psi^2}|_{\psi=0} \cos\beta \ , \ \mathcal{E}_\gamma(\theta) = \frac{\partial^2}{\partial\psi^2}|_{\psi=0} \cos\gamma \ , \ \mathcal{E}(\theta) = \frac{\partial^2}{\partial\phi^2}|_{\psi=0} Q \\ \omega_\beta = \arctan(x_1, y_0 + y_1) \quad , \quad \omega_\gamma = \arctan(x_1, -y_0 + y_1) \\ \Delta\omega_\beta = \arccos\dfrac{r_1^2 - x_0x_1 + y_0y_1}{r_0\sqrt{r_1^2 + y_0^2 + 2y_0y_1}} \ , \ \Delta\omega_\gamma = \arccos\dfrac{r_1^2 - x_0x_1 - y_0y_1}{r_0\sqrt{r_1^2 + y_0^2 - 2y_0y_1}} \\ \mathcal{T}_\beta(\theta) = r_0\sqrt{r_1^2 + y_0^2 + 2y_0y_1}\,\big[\cos\Delta\omega_\beta - \cos(\theta - \omega_\beta)\big] \\ \mathcal{T}_\gamma(\theta) = r_0\sqrt{r_1^2 + y_0^2 - 2y_0y_1}\,\big[\cos\Delta\omega_\gamma - \cos(\theta - \omega_\gamma)\big] \ . \end{cases}$$

Note that we are using the two-parameter arctan function here, the consequence of which is that

$$\begin{aligned} \cos\omega_\beta &= \frac{x_1}{\sqrt{y_0^2 + r_1^2 + 2y_0y_1}} \quad , \quad \sin\omega_\beta = \frac{y_0 + y_1}{\sqrt{y_0^2 + r_1^2 + 2y_0y_1}} \\ \cos\omega_\gamma &= \frac{x_1}{\sqrt{y_0^2 + r_1^2 - 2y_0y_1}} \quad , \quad \sin\omega_\gamma = \frac{-y_0 + y_1}{\sqrt{y_0^2 + r_1^2 - 2y_0y_1}} \quad . \end{aligned}$$

The significance of the omega angles is given be the next lemma, which is straightforward to check.

Lemma 3.22. *The signed angle (between $-\pi$ and π) subtended at C (B) between the ray from C (B) parallel to the positive x-axis and the ray $\overrightarrow{CA}$ ($\overrightarrow{BA}$) equals ω_β (ω_γ). Thus, when $x_1 < 0$, $\omega_\beta = \pi/2 + \angle C$ and $\omega_\gamma = \pi/2 + \angle B$. But, when $x_1 > 0$, $\omega_\beta = \pi/2 - \angle C$ and $\omega_\gamma = \pi/2 - \angle B$.*

The next lemma is clear from the definitions.

Lemma 3.23.

$$\mathcal{T}_\beta(\theta) = 0 \quad \Leftrightarrow \quad \theta = \omega_\beta \pm \Delta\omega_\beta \quad \textit{and} \quad \mathcal{T}_\gamma(\theta) = 0 \quad \Leftrightarrow \quad \theta = \omega_\gamma \pm \Delta\omega_\gamma.$$

The next couple lemmas also follow directly from the definitions, but require some effort.

Lemma 3.24.

$$\begin{cases} \mathcal{T}_\beta(\theta) = r_1^2 - x_0x_1 + y_0y_1 - r_0x_1\cos\theta - r_0(y_0+y_1)\sin\theta \\ \mathcal{T}_\gamma(\theta) = r_1^2 - x_0x_1 - y_0y_1 - r_0x_1\cos\theta + r_0(y_1-y_0)\sin\theta \\ \mathcal{E}_\beta(\theta) = \frac{\text{sign}\,(\theta_0-|\theta|)\,x_1}{s_1^3}\,\sin\frac{\theta_0-\theta}{2}\,\mathcal{T}_\beta(\theta) \\ \mathcal{E}_\gamma(\theta) = \frac{\text{sign}\,(\theta_0-|\theta|)\,x_1}{s_1^3}\,\sin\frac{\theta_0+\theta}{2}\,\mathcal{T}_\gamma(\theta) \\ \text{sign}[\mathcal{E}_\beta(\theta)] = \text{sign}(x_1)\,\text{sign}[\mathcal{T}_\beta(\theta)] \\ \text{sign}[\mathcal{E}_\gamma(\theta)] = \text{sign}(x_1)\,\text{sign}[\mathcal{T}_\gamma(\theta)]\,. \end{cases}$$

Lemma 3.25.

$$\begin{cases} \sin(\theta_0-\omega_\beta) & = & \frac{x_0y_0+x_0y_1+x_1y_0}{r_0\sqrt{r_1^2+y_0^2+2y_0y_1}} \\ \cos\Delta\omega_\beta - \cos(\omega_\beta-\theta_0) & = & \frac{r_1^2-y_0^2}{r_0\sqrt{r_1^2+y_0^2+2y_0y_1}} \\ \cos\Delta\omega_\beta - \cos(\omega_\beta+\theta_0) & = & \frac{\sqrt{r_1^2+y_0^2+2y_0y_1}}{r_0} \\ \cos\Delta\omega_\beta - \cos(\omega_\beta-\theta_\beta) & = & \frac{r_1^2-y_0^2-2x_0x_1}{r_0\sqrt{r_1^2+y_0^2-2y_0y_1}} \\ \cos\Delta\omega_\beta - \cos(\omega_\beta-\theta_\gamma) & = & \frac{(r_1^2-y_0^2)(r_1^2-y_0^2-2x_0x_1)}{r_0(r_1^2+y_0^2+2y_0y_1)\sqrt{r_1^2+y_0^2-2y_0y_1}} \end{cases}$$

Lemma 3.26. *The intersection point of the tangent line of the circle $\mathcal{C}$ at B, and the line $\overleftrightarrow{CX'}$ (X' being the center of the circle $\mathcal{C}'$) is on the circle that has $\overline{BC}$ as a diameter.*

Proof. By Lemma 3.2, the intersection point is found by solving the system

$$\begin{cases} y_0^2 + x_0x - y_0y = 0 \\ x_0y_0 + x_0y + y_0x = 0. \end{cases}$$

We get $(x,y) = (-2x_0y_0^2\,/\,(x_0^2+y_0^2)\,,\,y_0(y_0^2-x_0^2)\,/\,(x_0^2+y_0^2))$. The distance of this point from the origin is just y_0. Thus this point is on the circle that has $\overline{BC}$ as a diameter.

□

Direct computations reveal the following.

Lemma 3.27.

$$\begin{cases} \mathcal{T}_\beta(\theta_0) = r_1^2 - y_0^2 \ , \ \mathcal{T}_\gamma(\theta_0) = r_1^2 + y_0^2 - 2y_0y_1 \\ \mathcal{T}_\gamma(-\theta_0) = r_1^2 - y_0^2 \ , \ \mathcal{T}_\beta(-\theta_0) = r_1^2 + y_0^2 + 2y_0y_1 \\ \mathcal{T}_\beta(\theta_\beta) = r_1^2 - y_0^2 - 2x_0x_1 \ , \ \mathcal{T}_\gamma(\theta_\beta) = \dfrac{(r_1^2 - y_0^2)(r_1^2 - y_0^2 - 2x_0x_1)}{y_0^2 + r_1^2 + 2y_0y_1} \\ \mathcal{T}_\gamma(\theta_\gamma) = r_1^2 - y_0^2 - 2x_0x_1 \ , \ \mathcal{T}_\beta(\theta_\gamma) = \dfrac{(r_1^2 - y_0^2)(r_1^2 - y_0^2 - 2x_0x_1)}{y_0^2 + r_1^2 - 2y_0y_1} \ . \end{cases}$$

The following claims are immediately evident from Lemma 3.24.

Lemma 3.28. *The function $\mathcal{T}(\theta)$ is sinusoidal of period 2π, and as such, it has two roots over the interval $(-\pi, \pi]$. Furthermore, if $-\pi < \zeta < \eta < \pi$, and if $\mathcal{T}(\zeta)$ and $\mathcal{T}(\eta)$ are both nonzero and have the same sign, then the interval $[\zeta, \eta]$ either contains both or neither of the two roots.*

Here is another useful lemma, one whose proof appears to be a bit tricky.

Lemma 3.29. *When $x_1 > 0$,*

$$0 < \omega_\beta - \theta_\gamma < \pi \qquad \textit{and} \qquad 0 < \theta_\beta - \omega_\gamma < \pi.$$

Proof. To prove that $0 < \omega_\beta - \theta_\gamma < \pi$, we will consider both θ_β and θ_γ, and their corresponding points D and E on $\mathcal{C}$. See Figure 4. Because of the fact that $\angle C$ is acute, the angle θ_γ is constrained to be between $-\theta_0$ and $\pi - \theta_0$. Consider two cases: when $\theta_0 - \pi < \theta_\gamma < \pi - \theta_0$, and when $-\theta_0 < \theta_\gamma < \theta_0 - \pi$. Let's examine the first case. Fixing θ_γ and D, the angle ω_β is clearly greater that it would be if we allowed A to be on the circle $\mathcal{C}$, that is, if we allowed A to equal D. For a moment, suppose that $A = D$. Now consider varying θ_γ and D, with $\theta_0 - \pi < \theta_\gamma < \pi - \theta_0$, we observe two facts. If $\theta_\gamma \leqslant 0$, then certainly $\omega_\beta > \theta_\gamma$ since $\omega_\beta > 0$. But if $\theta_\gamma > 0$, then $\tan\theta_\gamma < \tan\omega_\beta$ because of the relative positions of C, X and D, specifically the fact that D is under $\overleftrightarrow{XC}$, which is positively sloped. The angles ω_β and θ_γ have absolute values less that $\pi/2$, and so, $0 < \omega_\beta - \theta_\gamma < \pi$, in the first case.

Now examine the second case, which, again is when $-\theta_0 < \theta_\gamma < \theta_0 - \pi$. Thus, D lies on the lower half of the $\mathcal{C}$. Clearly $\omega_\beta - \theta_\gamma > 0$ since $\omega_\beta > 0$ and $\theta_\gamma < 0$. We must however still show that $\omega_\beta - \theta_\gamma < \pi$. With θ_γ and D fixed, we will consider sliding the point A along the line $\overleftrightarrow{BD}$. Begin with A being the point of intersection of $\overleftrightarrow{BD}$ and the circle that has $\overline{BC}$ as a diameter. The points A, B and D are respectively (x_1, y_1), $(0, y_0)$ and $(x_0 + r_0\cos\theta_\beta, r_0\cos\theta_\beta)$. Thus, A being on $\overleftrightarrow{AC}$ requires that $x_1(r_0\sin\theta_\gamma - y_0) = (y_1 - y_0)(r_0\cos\theta_\gamma + x_0)$. In addition, A is on the circle with diameter $\overline{BD}$ if and only if $r_1 = y_0$. It can be checked that if A satisfies both of these conditions, then $x_1 = [r_0 + x_0 + (x_0 - r_0)t_\gamma^2][y_0^2 - 2r_0y_0t_\gamma + y_0^2t_\gamma^2] \ / \ \{\, r_0(1 + t_\gamma^2)[r_0 + x_0 - 2y_0t_\gamma + (r_0 - x_0)t_\gamma^2]\,\}$ and $y_0 + y_1 = [r_0 + x_0 + (x_0 - r_0)t_\gamma^2]^2\, y_0 \ / \ \{\, r_0(1 + t_\gamma^2)[r_0 + x_0 - 2y_0t_\gamma + (r_0 - x_0)t_\gamma^2]\,\}$, where $t_\gamma = \tan(\theta_\gamma/2) = \sin\theta_\gamma/(1 + \cos\theta_\gamma)$.

The slope of $\overleftrightarrow{AC}$ in this case is $\tan\omega_\beta = (y_0+y_1)/x_1 = [r_0+x_0+(x_0-r_0)t^2]$ / $[y_0 - 2r_0t + y_0t^2]$. The slope of $\overleftrightarrow{XD}$ is $\tan\theta_\gamma = 2t/(1-t^2)$.

Since $\omega_\beta = \pi/2 - \angle C < \pi/2$, we get $\omega_\beta - \theta_\gamma < \pi$ when $\theta_\gamma > -\pi/2$. Assume now instead that $-\pi < -\theta_0 < \theta_\gamma < -\pi/2$. Now, $(y_0+y_1)/x_1 - 2t/(1-t^2) = (1+t^2)[r_0+x_0-2y_0t+(r_0-x_0)t^2] \,/\, \{(1-t^2)[y_0-2r_0t+y_0t^2]\}$. This is negative, by the following reasoning: $r_0+x_0-2y_0t+(r_0-x_0)t^2 > 0$ because $y_0+y_1 > 0$ (because $|y_1| < y_0$), $y_0-2r_0t+y_0t^2 = 2(y_0-r_0\sin\theta_\gamma)/(1+\cos\theta_\gamma) > 0$, and $(1+t^2)/(1-t^2) = \sec\theta_\gamma < 0$. Thus $\tan\omega_\beta < \tan\theta_\gamma = \tan(\pi+\theta_\gamma)$. Since ω_β and $\pi+\theta_\gamma$ both lie between 0 and π, it follows that $\omega_\beta < \pi+\theta_\gamma$, so $\omega_\beta - \theta_\gamma < \pi$.

This accounts for the case when A lies on the intersection of the line $\overleftrightarrow{BD}$ and the circle with diameter $\overline{BD}$. Consider relocating A along $\overleftrightarrow{BD}$. It cannot be slid higher up since that would put it inside the circle, which is prohibited by the requirement that $\angle A$ be acute. If we instead instead slide it down $\overleftrightarrow{BD}$, then ω_β will decrease, but of course, θ_γ is unchanged. Thus the inequality $\omega_\beta - \theta_\gamma < \pi$ is maintained. This proves that $0 < \omega_\beta - \theta_\gamma < \pi$ in either case. Symmetric reasoning establishes that $0 < \theta_\beta - \omega_\gamma < \pi$.

□

The previous lemma can now be applied to prove the next lemma.

Lemma 3.30. *If $x_1 > 0$, then $\mathcal{T}(\theta)$ is (strictly) positive throughout the interval $[\theta_\gamma, \theta_\beta]$.*

Proof. From Lemmas 3.18 and 3.25, $\cos\Delta\omega_\beta < \cos(\omega_\beta - \theta_\gamma)$. Using Lemma 3.29, we can then deduce that $\Delta\omega_\beta > \omega_\beta - \theta_\gamma$, and, since $\omega_\beta > 0$ and $\Delta\omega_\beta < \pi$, we see that $-\pi < \omega_\beta - \Delta\omega_\beta < \theta_\gamma$. Thus, the function $\mathcal{T}(\theta)$ has a root in the interval $(-\pi, \pi]$ that is not in the interval $[\theta_\gamma, \theta_\beta]$. By Lemma 3.27, $\mathcal{T}(\theta_\beta) > 0$ and $\mathcal{T}(\theta_\gamma) > 0$. By Lemma 3.28, the interval $[\theta_\gamma, \theta_\beta]$ cannot contain any roots of $\mathcal{T}(\theta)$, and so $\mathcal{T}(\theta)$ must instead be positive throughout this interval.

□

The inequalities in Lemma 3.29 still hold when $x_1 < 0$.

Lemma 3.31. *When $x_1 < 0$, $0 < \omega_\beta - \theta_\gamma < \pi$ and $0 < \theta_\beta - \omega_\gamma < \pi$.*

Proof. See Figure 5. Since $\angle B$ is acute, the angle θ_γ is restricted to be between $\pi - \theta_0$ and θ_0, which are both positive. This corresponds to a point D on the upper half circle $\mathcal{C}$ whose coordinates are $(x_0 + r_0\cos\theta_\gamma, r_0\sin\theta_\gamma)$. By the definition of θ_γ, the point D is also on the line $\overleftrightarrow{AB}$. Also on the upper half of $\mathcal{C}$ is a point P such that the tangent line to $\mathcal{C}$ at P is parallel to the line $\overleftrightarrow{BD}$. Clearly P lies along the subarc of $\mathcal{A}$ (in the upper half plane) connecting B and D.

Let θ_P be the angle between θ_γ and θ_0 such that P has coordinates $(x_0 + r_0\cos\theta_P, r_0\sin\theta_P)$. Let M be the midpoint of the segment $\overline{BC}$. It is now straightforward to check that $\pi - \theta_B = \angle MXP = \angle CBA = \angle B$. (Again,

X denotes the center of $\mathcal{C}$.) To see that $\angle MXP = \angle CBA$, apply a 90-degree rotation. So, $\theta_P = \pi - \angle B > \theta_\gamma$. So, $\omega_\beta - \theta_\gamma > (\pi/2 + \angle C) + (\angle B - \pi) = \angle B + \angle C - \pi/2 = \pi/2 - \angle A > 0$. Also, $\omega_\beta - \theta_\gamma < \omega_\beta = \pi/2 + \angle C < \pi$.

This proves that $0 < \omega_\beta - \theta_\gamma < \pi$. Symmetric reasoning establishes that $0 < \theta_\beta - \omega_\gamma < \pi$.

□

We are now prepared to show claims about the sign of $\mathcal{T}(\theta)$ over intervals of interest.

Lemma 3.32. *Assume that* $x_1 < 0$.

$$\begin{cases} \textit{Case 1.} & |\theta_\beta| < \theta_0 \wedge |\theta_\gamma| < \theta_0 \;\Rightarrow \\ & \mathcal{T}(\theta) \textit{ is (strictly) positive throughout the interval } [\theta_\beta, \theta_\gamma]; \\ \textit{Case 2.} & |\theta_\beta| < \theta_0 \wedge |\theta_\gamma| > \theta_0 \;\Rightarrow \\ & \mathcal{T}(\theta) \textit{ is (strictly) positive throughout the interval } [\theta_\beta, \theta_0]; \\ \textit{Case 3.} & |\theta_\beta| > \theta_0 \wedge |\theta_\gamma| < \theta_0 \;\Rightarrow \\ & \mathcal{T}(\theta) \textit{ is (strictly) positive throughout the interval } [-\theta_0, \theta_\gamma]; \\ \textit{Case 4.} & |\theta_\beta| > \theta_0 \wedge |\theta_\gamma| > \theta_0 \;\Rightarrow \\ & \mathcal{T}(\theta) \textit{ is (strictly) positive throughout the interval } [-\theta_0, \theta_0]. \end{cases}$$

Proof. First, consider Case 1 and assume its hypothesis. From Lemma 3.25, $\cos \Delta\omega_\beta > \cos(\omega_\beta - \theta_\gamma)$. Using Lemma 3.31, we can then deduce that $\Delta\omega_\beta < \omega_\beta - \theta_\gamma$, and, since $\omega_\beta < \pi$ and $\Delta\omega_\beta > 0$, we see that $\theta_\gamma < \omega_\beta - \Delta\omega_\beta < \pi$. Thus, the function $\mathcal{T}_\beta(\theta)$ has a root in the interval $(-\pi, \pi]$ that is not in the interval $[\theta_\beta, \theta_\gamma]$. By Lemma 3.27, $\mathcal{T}_\beta(\theta_\beta) > 0$ and $\mathcal{T}_\beta(\theta_\gamma) > 0$. By Lemma 3.28, the interval $[\theta_\beta, \theta_\gamma]$ cannot contain any roots of $\mathcal{T}_\beta(\theta)$, and so $\mathcal{T}_\beta(\theta)$ must instead be positive throughout this interval. By symmetry, $\mathcal{T}_\gamma(\theta)$ must instead be positive throughout this interval. Therefore, $\mathcal{T}(\theta)$ must instead be positive throughout this interval.

Next, consider Case 2 and assume its hypothesis. From Lemma 3.25, $\cos \Delta\omega_\beta > \cos(\omega_\beta - \theta_0)$. Since $\pi/2 < \omega_\beta < \pi$ and $\pi/2 < \theta_0 < \pi$, we get $0 < \omega_\beta - \theta_0 < \pi/2$. It follows that $\Delta\omega_\beta < \omega_\beta - \theta_0$. Since $\omega_\beta < \pi$ and $\Delta\omega_\beta > 0$, we see that $\theta_0 < \omega_\beta - \Delta\omega_\beta < \pi$. Thus, the function $\mathcal{T}_\beta(\theta)$ has a root in the interval $(-\pi, \pi]$ that is not in the interval $[\theta_\beta, \theta_0]$. By Lemma 3.27, $\mathcal{T}_\beta(\theta_\beta) > 0$ and $\mathcal{T}_\beta(\theta_0) > 0$. By Lemma 3.28, the interval $[\theta_\beta, \theta_0]$ cannot contain any roots of $\mathcal{T}_\beta(\theta)$, and so $\mathcal{T}_\beta(\theta)$ must instead be positive throughout this interval.

Also from Lemma 3.25, $\cos \Delta\omega_\gamma > \cos(\theta_\beta - \omega_\gamma)$. By Lemma 3.31, we get $\Delta\omega_\gamma < \theta_\beta - \omega_\gamma$. Since $-\pi/2 < \omega_\gamma$ and $\Delta\omega_\gamma > 0$, we get $-\pi < \omega_\gamma + \Delta\omega_\gamma < \theta_\beta$. Thus, the function $\mathcal{T}_\gamma(\theta)$ has a root in the interval $(-\pi, \pi]$ that is not in the interval $[\theta_\beta, \theta_0]$. By Lemma 3.27, $\mathcal{T}_\gamma(\theta_\beta) > 0$ and $\mathcal{T}_\gamma(\theta_0) > 0$. By Lemma 3.28, the interval $[\theta_\beta, \theta_0]$ cannot contain any roots of $\mathcal{T}_\gamma(\theta)$, and so $\mathcal{T}_\gamma(\theta)$ must instead be positive throughout this interval. Since, $\mathcal{T}_\beta(\theta)$ is also positive throughout this interval, we find that $\mathcal{T}(\theta)$ must also be positive throughout this interval.

Case 3 can be handled by a symmetric argument based on Case 2. Case 4 follows in a manner that is similar to the previous cases.

□

At last, we arrive at the principal goal of this section, namely, stating and proving the following result.

Theorem 3.33. *Apart from the apexes B and C of the toroid $\mathcal{T}or_\alpha$, the quantity $Q = \mu \cos\beta + \nu \sin\gamma$ (for fixed $\mu, \nu > 0$) has either one or two critical points on $\mathcal{T}or_\alpha$ that also lie in the xy-plane. At these points, Q has a local maximum value.*

Proof. The intersection of $\mathcal{T}or_\alpha$ and the xy-plane consists of two circular arcs, as discuessed earlier. Without loss of generality, we may assume that A has coordinates (x_1, y_1) with $x_1 > 0$. Along the right arc $\mathcal{A}$ (part of $\mathcal{C}$) there is exactly one value $\tilde{\theta}$ with $|\tilde{\theta}| < \theta_0$ and $\mathcal{S}(\tilde{\theta}) = 0$, and so $\mathcal{D}(\tilde{\theta}) = 0$ too. At the corresponding point $(x_0 + r_0\cos\tilde{\theta}, r_0\sin\tilde{\theta})$, the quantity Q has a critical value on $\mathcal{T}or_\alpha$. Here, $\mathcal{D}(\tilde{\theta})$ is decreasing, $\mathcal{D}'(\tilde{\theta}) < 0$ and $\mathcal{E}(\tilde{\theta}) < 0$. Direct computation reveals that when ψ is a multiple of π, $\partial^2 Q/\partial\psi\partial\theta = 0$. Consequently, at the critical point, the Hessian determinant of Q with respect to θ and ψ equals

$$\begin{vmatrix} \mathcal{D}'(\tilde{\theta}) & 0 \\ 0 & \mathcal{E}(\tilde{\theta}) \end{vmatrix} = \mathcal{D}'(\tilde{\theta})\,\mathcal{E}(\tilde{\theta}) > 0.$$

It follows that Q has a relative maximum value at this point, on $\mathcal{T}or_\alpha$.

The left arc $\mathcal{A}'$ (part of $\mathcal{C}'$) can be analyzed by first reflecting the xy-plane about the y-axis. This means we are now assuming that $x_1 < 0$ and are examining the right arc $\mathcal{A}$ again. By the foregoing analysis, the situation is the same as before except that there might not be a value of θ with $|\theta| < \theta_0$ and $\mathcal{S}(\theta) = 0$. If there is, then it is unique, and also corresponds to a relative maximum for Q on $\mathcal{T}or_\alpha$, by the same reasoning as before. There can be no other critical points for Q on $\mathcal{T}or_\alpha$ in the xy-plane, other than the apexes of the toroid.

□

4. Additional Singularities not in the xy-plane

Our focus now shifts to the upper half of the toroid $\mathcal{T}or_\alpha$, that is, the points where $z > 0$. Remember that both $\mathcal{T}or_\alpha$ and the scalar field $Q = \mu\cos\beta + \nu\cos\gamma$ on it are symmetric under reflection about the xy-plane ($z \to -z$). We will see that surface gradient $\overrightarrow{\nabla}_\alpha Q$ (defined in (1.1)), on the upper half of $\mathcal{T}or_\alpha$, can have at most one singularity, and if it exists, this point cannot correspond to a relative extreme for Q on $\mathcal{T}or_\alpha$.

In this section, it will prove helpful to use a different coordinate system than the one used in the provious sectiom, though we continue to assume that the triangle $\triangle ABC$ lies in the xy-plane. However, without loss of generality, we will now assume that all of its vertices lie on the unit circle in the xy-plane. Moreover, letting $(x_j, y_j) = (\cos\theta_j, \sin\theta_j)$ $(j = 1, 2, 3)$ denote the coordinates of A, B and C, respectively, we will further assume that $\theta_1 + \theta_2 + \theta_3 = 0$. By simply scaling, translating and/or rotating the coordinate system previously used, we can easy obtain a coordinate system of this sort. It will also be helpful to set $t_j = \tan(\theta_j/2)$ $(j = 1, 2, 3)$, and to notice that $\cos\theta_j = (1-t_j^2)/(1+t_j^2)$, $\sin\theta_j = 2t_j/(1+t_j^2)$ and $t_1t_2t_3 = t_1 + t_2 + t_3$.

Now, in order for $\vec{\bigtriangledown}_\alpha Q$ to vanish at a point on $\mathcal{Tor}_\alpha$, it is necessary and sufficient that $\vec{\bigtriangledown}\cos\alpha \times \vec{\bigtriangledown} Q = 0$ at that point. This means that $\vec{\bigtriangledown}\cos\alpha$ and $\vec{\bigtriangledown} Q$ are linearly dependent. We will now study the equation

$$\lambda \vec{\bigtriangledown}\cos\alpha + \mu \vec{\bigtriangledown}\cos\beta + \nu \vec{\bigtriangledown}\cos\gamma = 0, \tag{4.1}$$

usually treating λ, μ and ν all as variables. Later, we will pause to remember that we are ultimately interested in whether or not a suitable value of λ exists for given values of μ and ν. Equation (4.1) can be rewritten as follows.

Lemma 4.1.

$$[\lambda\ \mu\ \nu] \begin{bmatrix} 0 & \frac{s_3^2 - s_2^2 - a^2}{2s_2^2 s_3} & \frac{s_2^2 - s_3^2 - a^2}{2s_2 s_3^2} \\ \frac{s_3^2 - s_1^2 - b^2}{2s_1^2 s_3} & 0 & \frac{s_1^2 - s_3^2 - b^2}{2s_1 s_3^2} \\ \frac{s_2^2 - s_1^2 - c^2}{2s_1^2 s_2} & \frac{s_1^2 - s_2^2 - c^2}{2s_1 s_2^2} & 0 \end{bmatrix} = [0\ 0\ 0]. \tag{4.2}$$

Proof. The following can be checked directly using the definitions in Section 2.

$$[\lambda\ \mu\ \nu] \begin{bmatrix} 0 & \frac{s_3^2 - s_2^2 - a^2}{2s_2^2 s_3} & \frac{s_2^2 - s_3^2 - a^2}{2s_2 s_3^2} \\ \frac{s_3^2 - s_1^2 - b^2}{2s_1^2 s_3} & 0 & \frac{s_1^2 - s_3^2 - b^2}{2s_1 s_3^2} \\ \frac{s_2^2 - s_1^2 - c^2}{2s_1^2 s_2} & \frac{s_1^2 - s_2^2 - c^2}{2s_1 s_2^2} & 0 \end{bmatrix} \begin{bmatrix} \widehat{s_1} \\ \widehat{s_2} \\ \widehat{s_3} \end{bmatrix} = \vec{0}.$$

Since the vectors $\widehat{s_1}$, $\widehat{s_2}$ and $\widehat{s_3}$ are linearly independent, (4.2) follows. □

The next result is interesting and identifies clearly where nontrivial solutions to (4.1) are possible.

Lemma 4.2. *The determinant of the* 3×3 *matrix in (4.2) is*

$$\frac{64(t_1 - t_2)^2(t_2 - t_3)^2(t_3 - t_1)^2}{4s_1^2 s_2^2 s_3^2(1 + t_1^2)(1 + t_2^2)(1 + t_3^2)} (x^2 + y^2 - 1).$$

Consequently, a solution to (4.1) (or (4.2)) is only possible at points on the cylinder defined by

$$x^2 + y^2 = 1.$$

Proof. Direct computation reveals that the discriminant equals $[a^2(s_1^2-s_2^2)(s_1^2-s_3^2)+b^2(s_2^2-s_3^2)(s_2^2-s_1^2)+c^2(s_3^2-s_1^2)(s_3^2-s_2^2)]/[4s_1^3s_2^3s_3^3]$. This can be expanded in terms of x, y, z, x_j and y_j $(j = 1, 2, 3)$, and then expanded further in terms of t_j $(j = 1, 2, 3)$. The above formula results from simplifying this. ◻

Generally, each point on the cylinder $x^2 + y^2 = 1$ admits a unique homogeneous triple $(\lambda : \mu : \nu)$ that satifies (4.1) (and (4.2)) at that point. In this way, it is useful to consider the function on the cylinder that assigns this homogeneous triple. We will also define $\rho = \nu/\mu$ and treat this too as a function defines on the cylinder, a function that is allowed to have ∞ as a value, so it is a function whose codomain is the real projective line.

In order to unltimately resolve the issue of singularities of $\vec{\nabla}_\alpha Q$ on $\mathcal{Tor}_\alpha$, it will prove useful to ignore these for a while, and instead focus on two functions defined on upper half of the cylinder $x^2 + y^2 = 1$, where $z > 0$. Let us denote this half cylinder by $\mathcal{H}$. The functions of interest on $\mathcal{H}$ are ρ and $\cos\alpha$. Letting the coordinates of a points on $\mathcal{H}$ be $(x, y, z) = (\cos\theta, \sin\theta, z)$, it will also be handy to set $t = \tan(\theta/2)$, so that $\cos\theta = (1-t^2)/(1+t^2)$ and $\sin\theta = 2t/(1+t^2)$.

Lemma 4.3. *At a point $\mathcal{H}$, we have* $(\lambda : \mu : \nu) =$

$$\begin{aligned}
\Big(\quad & (1+t_1t)\,/\,[(t_2-t_3)\sqrt{1+t_1^2}\sqrt{4(t-t_1)^2+(1+t_1^2)(1+t^2)z^2}\,] \;:\\
& (1+t_2t)\,/\,[(t_3-t_1)\sqrt{1+t_2^2}\sqrt{4(t-t_2)^2+(1+t_2^2)(1+t^2)z^2}\,] \;:\\
& (1+t_3t)\,/\,[(t_1-t_2)\sqrt{1+t_3^2}\sqrt{4(t-t_3)^2+(1+t_3^2)(1+t^2)z^2}\,] \quad\Big).
\end{aligned}$$

Thus, $\rho =$

$$-\frac{\sqrt{1+t_2^2}\,(t_1-t_3)(1+t_3t)\,\sqrt{4(t-t_2)^2+(1+t_2^2)(1+t^2)z^2}}{\sqrt{1+t_3^2}\,(t_1-t_2)(1+t_2t)\,\sqrt{4(t-t_3)^2+(1+t_3^2)(1+t^2)z^2}}\,.$$

Proof. Equation 4.2 can be rewritten as follows:

$$[\lambda\ \mu\ \nu]\begin{bmatrix}\frac{1}{s_2s_3} & 0 & 0\\ 0 & \frac{1}{s_3s_1} & 0\\ 0 & 0 & \frac{1}{s_1s_2}\end{bmatrix}\begin{bmatrix}0 & s_3\cos\alpha-s_2 & s_2\cos\alpha-s_3\\ s_3\cos\beta-s_1 & 0 & s_1\cos\beta-s_3\\ s_2\cos\gamma-s_1 & s_1\cos\gamma-s_2 & 0\end{bmatrix}$$

$= [0\ 0\ 0]$. Setting $\lambda' = \lambda/(s_2s_3)$, $\mu' = \mu/(s_3s_1)$ and $\nu' = \nu/(s_1s_2)$, we see that $\mu'(s_3\cos\beta - s_1) + \nu'(s_2\cos\gamma - s_1) = 0$. So, $\rho = (s_2\nu')/(s_3\mu') = -[s_2(s_3\cos\beta - s_1)]\,/\,[s_3(s_2\cos\gamma - s_1)] = -[s_2(s_3^2-s_1^2-b^2)]\,/\,[s_3(s_2^2-s_1^2-c^2)]$. This can be expanded to obtain the formula in the lemma.

◻

The next formula is established by writing $\cos^2\alpha$ as $(s_2^2+s_3^2-a^2)^2/(4s_2^2s_3^2)$, and then expanding as before, and simplifying.

Lemma 4.4.

$$\cos\alpha = \left[4(1+t_2t_3)(t-t_2)(t-t_3) + (1+t_2^2)(1+t_3^2)(1+t^2)z^2\right] / \left[\sqrt{1+t_2^2}\right.$$
$$\left.\cdot\sqrt{1+t_3^2}\,\sqrt{4(t-t_2)^2+(1+t_2^2)(1+t^2)z^2}\,\sqrt{4(t-t_3)^2+(1+t_3^2)(1+t^2)z^2}\right].$$

The quantities ρ^2 and $\cos^2\alpha$ can be treated as functions of t and Z, where $Z = z^2$. As such, their partial derivatives are readily computed. Define:

$$\left.\frac{dZ}{dt}\right|_\rho = -\frac{\partial(\rho^2)/\partial t}{\partial(\rho^2)/\partial Z} \quad \text{and} \quad \left.\frac{dZ}{dt}\right|_\alpha = -\frac{\partial(\cos^2\alpha)/\partial t}{\partial(\cos^2\alpha)/\partial Z}.$$

These are the relative rates of change of Z and t along a curve of constant ρ, and along a curve of constant α, respectively. The following facts can be immediately verified.

Lemma 4.5.

$$\left.\frac{dZ}{dt}\right|_\rho = \left[(Z+4)[(1+t_2^2)(1+t_3^2)(1+t^2)^2Z - 4(t-t_2)(t-t_3)((t_2t_3-1)\cdot\right.$$
$$\left.(t^2-1)+2(t_2+t_3)t)]\right] / \left[2(1+t^2)(1+t_2t)(1+t_3t)[(t_2+t_3)(t^2-1)+2(1-t_2t_3)t]\right],$$
$$\left.\frac{dZ}{dt}\right|_\alpha = \left[8(1+t_2t)(1+t_3t)[(t_2+t_3)(1-t^2)+2(t_2t_3-1)t]Z\right] / \left[(1+t^2)\cdot\right.$$
$$\left.[(1+t_2^2)(1+t_3^2)(1+t^2)^2Z - 4(t-t_2)(t-t_3)((t_2t_3-1)(t^2-1)+2(t_2+t_3)t)]\right],$$

$$\textit{and} \quad \left.\frac{dZ}{dt}\right|_\rho \left.\frac{dZ}{dt}\right|_\alpha = \frac{-4Z(Z+4)}{(1+t^2)^2} < 0.$$

An immediate consequence of the surprisingly simple last equation is the following:

Lemma 4.6. *A constant-α curve and a constant-ρ curve, on $\mathcal{H}$, are never tangent to each other at a point.*

The next two lemmas follow by examining the numerators and denominators of $dZ/dt\,|_\rho$ and $dZ/dt\,|_\alpha$.

Lemma 4.7. *For each of $t = -1/t_2$, $t = -1/t_3$ and $t = (1 \pm \sqrt{1+t_1^2})/t_1$, there is a vertical constant-ρ curve. Constant-α curves have horizontal tangent lines at points where they cross these lines. Moreover, other constant-ρ curves have no vertical tangent lines, and constant-α curves do not have horizontal tangent lines at any other points.*

Lemma 4.8. *A constant-ρ curve has a horizontal tangent line, and a constant-α curve has a vertical tangent line at points of intersection of such curves with the curve Γ described by $(1+t_2^2)(1+t_3^2)(1+t^2)^2\,Z = 4(t-t_2)(t-t_3)\,[(t_2t_3-1)(t^2-1)+2(t_2+t_3)\,t]$, and nowhere else.*

The curve Γ will actually not affect the situation we are ultimately focused on because of the following fact.

Lemma 4.9. *At any point on Γ, $\alpha > \angle A$.*

Proof. Assume this is false, and consider a point on Γ where $\alpha \leqslant \angle A < \pi/2$. Solving the equation in Lemma 4.8 for Z, and substituting this into the formula for $\cos^2\alpha$ essentially given in Lemma 4.4, we obtain

$$\frac{4(t-t_2)(t-t_3)(1+t_2t)(1+t_3t)}{[\,(t_2+t_3)(t^2-1)+2(1-t_2t_3)t\,]^2}\,.$$

Now, $\cos^2\angle A = (b^2+c^2-a^2)^2/(2bc)^2 = (1+t_2t_3)^2\,/\,[\,(1+t_2^2)(1+t_3^2)\,]$. From these formulas, we obtain $\cos^2\alpha - \cos^2\angle A =$

$$-\,\frac{(t_2-t_3)^2\,[\,(t_2t_3-1)(t^2-1)+2(t_2+t_3)t\,]^2}{(1+t_2^2)(1+t_3^2)\,[\,(t_2+t_3)(t^2-1)+2(1-t_2t_3)\,]^2}\;<\;0.$$

Hence $\cos\alpha < \cos\angle A$, since both angles are in the range from 0 to $\pi/2$. We conclude that $\alpha > \angle A$. This is so for any point on Γ.

□

Only the portion $\mathcal{H}_0$ of $\mathcal{H}$ for which $\alpha \leqslant \angle A$ will be of further interest. This consists of all but a bounded portion of $\mathcal{H}$ near the xy-plane. The next lemma is now automatic.

Lemma 4.10. *On the curve Γ, Z varies as a function of t (or equivalently θ). When $Z \geqslant 0$, this implies a unique value of $z \geqslant 0$, but when $Z < 0$, there is no corresponding real value of z. Also, none of the curve Γ on lies on $\mathcal{H}_0$.*

To better understand the constant-ρ curves and constant-α curves on $\mathcal{H}_0$, consider the next two lemmas. The first lemma is just obtained from Lemma 4.3 by direct computation.

Lemma 4.11. *Working on $\mathcal{H}_0$ and keeping t fixed (so θ fixed), the limit as $z \to \infty$ of ρ equals*

$$\rho_\infty \;=\; -\,\frac{(1+t_2^2)(t_2t_3^2-t_2-2t_3)(1+t_3t)}{(1+t_3^2)(t_2^2t_3-2t_2-t_3)(1+t_2t)}\,.$$

By inverting this, we see that any given value of ρ_∞ corresponds to a unique value of t equal to

$$-\,\frac{(1+t_2^2)(t_2t_3^2-t_2-2t_3)+(1+t_3^2)(t_2^2t_3-2t_2-t_3)\,\rho_\infty}{t_3(1+t_2^2)(t_2t_3^2-t_2-2t_3)+t_2(1+t_3^2)(t_2^2t_3-2t_2-t_3)\,\rho_\infty}\,.$$

Lemma 4.12. *Apart from the vertical lines identified in Lemma 4.7, a constant-ρ curve on $\mathcal{H}_0$ is such that Z varies monotonically as a smooth function of t (or θ), defined over a certain interval of t (depending on the value of ρ). A constant-α curve on $\mathcal{H}_0$ is such that Z varies as a smooth function of t (or θ) for all $t \in (-\infty, \infty]$ (or $\theta \in (-\pi, \pi]$). Moreover, constant-ρ curves and constant-α curves are connected curves.*

Proof. $dZ/dt|_\rho$ is never zero on $\mathcal{H}_0$, and except on the special vertical lines, it is smoothly defined and not infinite, so its sign does not change along a constant-ρ curve. This establishes the monotonicity claim on a constant-ρ curve. The fact that this curve is connected and unbounded follows from Lemma 4.11.

Concerning a constant-α curve, notice that since $\alpha < \angle A$, the intersection of $\mathcal{T}or_\alpha$ and the xy-plane consists of two circular arcs, $\mathcal{A}$ and $\mathcal{A}'$, connecting B and C that together enclose the unit circle in the xy-plane. Consider generating $\mathcal{T}or_\alpha$ by revolving either of these arcs about the line $\overleftrightarrow{BC}$. It is immediately clear that this will intersect $\mathcal{H}_0$ in a closed curve that wraps one time around $\mathcal{H}_0$. The claims concerning a constant-α curve follow from this fact.

□

It is now straightforward to prove the next lemma.

Lemma 4.13. *A constant-α curve on $\mathcal{H}_0$ does not contain two points that have the same value of ρ.*

Proof. Assume this is false, and let $t_L < t_R$ be such that the two point on the curve corresponding to $t = t_L$ and to $t = t_R$ have the same value of ρ. By Lemma 4.11, there is a corresponding point on the curve for each t in the closed interval $[t_L, t_R]$, and ρ varies smoothly along this interval. By Rolle's Theorem, there exists some $\tilde{t}$ with $t_L < \tilde{t} < t_R$ such that $d\rho/dt\,|_\alpha = 0$. At the corresponding point on the constant-α curve, this curve must be tangent to a constant-ρ curve, but Lemma 4.6 prohibits this, a contradiction.

□

We finally arrive at the main goal of this section, namely the establishment of the next result. We return to considering fixed values of α, μ and ν.

Theorem 4.14. *If a singular point for the vector field $\overrightarrow{\nabla}_\alpha Q$ on $\mathcal{T}or_\alpha$ occurs on the upper half of $\mathcal{T}or_\alpha$ ($z > 0$, $\alpha < \angle A$), then it is unique and occurs on the intersection of $\mathcal{T}or_\alpha$ and the half cylinder $\mathcal{H}$. Moreover, it cannot be a relative extremun point for Q*

Therefore, the supremum value for Q on $\mathcal{T}or_\alpha$ must occur either when approaching B or C from some direction, or at another relative extremum point in the xy-plane. The infimum value for Q on $\mathcal{T}or_\alpha$ must always occur when approaching either B or C from some direction.

Proof. Such a singular point only occurs when $\vec{\nabla}\cos\alpha \times \vec{\nabla}Q = 0$. This in turn requires that $\lambda \vec{\nabla}\cos\alpha + \mu \vec{\nabla}\cos\beta + \nu \vec{\nabla}\cos\gamma = 0$ for some value of λ. Lemma 4.2 shows that this can only happen on $\mathcal{H}$. Since $\alpha \leqslant \angle A$, it can only happen on $\mathcal{H}_0$, by definition. Lemma 4.12 establishes that such a singular point on the intersection of $\mathcal{T}or_\alpha$ and $\mathcal{H}_0$ must be unique.

Consider modifying $\mathcal{T}or_\alpha$ in a very small way, as follows. Begin with a very small circle on $\mathcal{T}or_\alpha$, near the apex B and around $\overleftrightarrow{BC}$, of the sort used in Lemma 2.9, one on which the surface gradient $\vec{\nabla}_\alpha Q$ does not vanish, and has the winding number 0, 1 or 2. This circle divides $\mathcal{T}or_\alpha$ into two portions, one of which is tiny and includes B. Now alter $\mathcal{T}or_\alpha$ by smoothly replacing this tiny portion with a smooth tiny region. Do likewise near the apex C using a tiny circle near it. The result is a smooth surface on which the scalar field Q and the vector field $\vec{\nabla}_\alpha Q$ are smoothly defined.

Since the winding number of $\vec{\nabla}_\alpha Q$ around the tiny circle near B is still 0, 1 or 2, the sum of the indexes of the singularities inside the new tiny region near B must be 0, 1 or 2. Similarly near C. By Theorem 3.33, there are either one or two other singuarities for $\vec{\nabla}_\alpha Q$ on (the slightly altered) $\mathcal{T}or_\alpha$ that occur in the xy-plane, and these are relative maximum points for Q. So the index of each of these, as a singularity for $\vec{\nabla}_\alpha Q$, is one. Together the sum of the indexes of all of the singularities considered thus far is at least one.

Now, if there is a singularity for $\vec{\nabla}_\alpha Q$ in the upper half of (the slightly altered) $\mathcal{T}or_\alpha$, then it is unique, and by reflection, there is a corresponding singularity in the lower half of (the slightly altered) $\mathcal{T}or_\alpha$, and vice-versa. The sum of the indexes of these two singularities must be an even integer. However, it cannot exceed zero because, by the Poincaré-Hopf Theorem (see [2]), the sum of the indexes of all of the singularities of $\vec{\nabla}_\alpha Q$ on the slightly altered $\mathcal{T}or_\alpha$ must equal two. This is because this surface is smooth and homeomorphic to a sphere. Thus the index of a singularity $\vec{\nabla}_\alpha Q$ in the upper half of (the slightly altered) $\mathcal{T}or_\alpha$ cannot be positive, and so cannot correspond to a relative extremum for Q. Ditto for the lower half of the slightly altered $\mathcal{T}or_\alpha$. The remaining claims now follow easily.

□

5. Completion of the Proof of Theorem 1.1 and Beyond

Proving the remaining parts of Theorem 1.1 will depend on the preceding analysis concerning the scalar field $Q = \mu\cos\beta + \nu\cos\gamma$ on the toroid $\mathcal{T}or_\alpha$ ($\alpha < \angle A$). These last two parts are actually easy consequences of the following more general assertion.

Theorem 5.1. *Assuming that $\triangle ABC$ is acute, that $\alpha < \angle A$, and that μ and ν are non-negative constants, the inequality $Q = \mu\cos\beta + \nu\cos\gamma \geqslant$*

$\min\{\,\mu\cos\angle B+\nu\cos(\angle B+\alpha)\,,\,\mu\cos(\angle C+\alpha)+\nu\cos\angle C\,\}$ *holds at all points on* $\mathcal{T}or_\alpha$*, other than* B *and* C*.*

Proof. First, consider Q with $\mu = 1$ and $\nu = 0$. By Theorem 4.14, the infimum value for Q ($= \cos\beta$) occurs when approaching either B or C. The limiting value of Q when approaching B is just $\cos\angle B$, independent of the path used to approach B. The situation is more complicated near C. By again considering a tiny circle on $\mathcal{T}or_\alpha$, near C and around $\overleftrightarrow{BC}$, similar to the circle used in Lemma 2.9 near B, it is straightforward to check that the extreme values for β occur at the two points of intersection of the circle and the xy-plane. These extreme values for β, when infinitesimally close to C, are $\angle C + \alpha$ and $|\angle C - \alpha|$. Thus, $\cos(\angle C + \alpha)$ is the minimal limiting value for $\cos\beta$ when approaching C on $\mathcal{T}or_\alpha$. The inequality to be proved here therefore holds when $\mu = 1$ and $\nu = 0$. It similarly holds when $\mu = 0$ and $\nu = 1$.

In the general case, Theorem 4.14 still guarantees that the infimum value for Q on $\mathcal{T}or_\alpha$ occurs when approaching either B or C. The extreme values for Q on the infinitesimally small circles on $\mathcal{T}or_\alpha$ around B and around C are easily determined from the above special cases. The specified inequality in the general case then follows directly.

□

We are now prepared to complete the proof of Theorem 1.1.

Proof of Part 3 of Theorem 1.1. This is just the $\mu = 1$ and $\nu = 0$ special case of Theorem 5.1, since all of the cosines involved are of angles in the range from 0 to π. This establishes Part 3 of Theorem 1.1.

□

Proof of Part 4 of Theorem 1.1. Consider the special case of Theorem 5.1 where $\mu = \cos\angle C$ and $\nu = \cos\angle B$. Here we obtain $Q \geqslant \min\{\,\cos\angle C\cos\angle B + \cos\angle B\cos(\angle B+\alpha)\,,\,\cos\angle C\cos(\angle C+\alpha)+\cos\angle B\cos\angle C\,\} \geqslant \min\{\,\cos\angle C\cos\angle B+\cos\angle B\cos(\angle B+\angle A)\,,\,\cos\angle C\cos(\angle C+\angle A)+\cos\angle B\cos\angle C\,\} = \min\{0,0\} = 0$. This establishes Part 4 of Theorem 1.1.

□

Note: If we regard the constaints in Theorem 1.1 as describing a region in the 3-dimensional real space of all possible triples (α,β,γ), this region is bounded. It is "nearly" a non-convex polyhedron, typically. Figure 6 shows an example. The last set of constraints in Theorem 1.1 are nonlinear, which is why the region fails to be a polyhedron.

Note: The problem considered in this paper can be flipped. Instead of asking what restrictions given A, B and C place on α, β and γ, we could instead ask what restictions given α, β and γ place on A, B and C. This problem is also

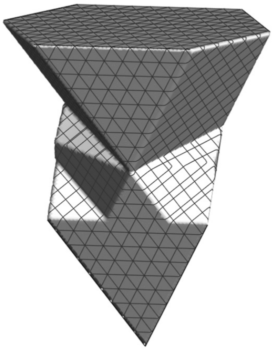

FIGURE 6. Bounding region

non-trivial, though it is simpler. The interested reader is encouraged to explore it via an evident system involving six applications of the Law of Cosines.

Note: While the possible triples (α, β, γ) have been of primary interest, a related issue is constraining the possible triples of dihedral angles (angles between faces of the tetrahedron) at the point P. Three equations involving these, the other three dihedral angles (involving the face ΔABC), and the known interior angles of ΔABC, can be helpful for this. Still, useful constraints here seem to be at least as complicated as the constrains for the (α, β, γ) triples.

Note: Besides the set of constraints in Theorem 1, some "near constraints" have also been discovered:

$$\alpha \leqslant \angle A \ \wedge \ \angle B \geqslant \angle C \ \wedge \ \angle B \leqslant \angle A + \angle C \ \rightarrow$$
$$(\angle A)(\beta + \gamma - \alpha) + (\angle B - \angle C)(\alpha + \beta - \gamma) \leqslant 2\angle A \angle B,$$

and

$$\alpha \leqslant \angle A \ \wedge \ \angle B \geqslant \angle C \ \wedge \ \beta \geqslant \angle B \ \rightarrow$$
$$(\angle A)(\beta + \gamma - \alpha) + (\angle B - \angle C)(\beta - \gamma - \alpha) \leqslant 2\angle A \angle C,$$

and suitable permutations of these. While these appear to further restrict the region of allowable (α, β, γ) triples, they have not been rigourously proved to do so. In fact, there is some evidence that they fail very slightly but only very near the triangle vertices, A, B and C. Though evidence has been gathered, for different triangles, these claims are currently only conjectural.

Note: The appendix contains the C++ source code for a computer program that can be used to explore and test how well the constraints in Theorem 1.1 perform in describing the region of all allowable (α, β, γ) triples, as a subset of the cube $[0, \pi]^3$, for specified values of A, B and C. It is also available at https://github.com/mqrieck/tetrahedron_test.cpp. Besides gathering data, it can also be used to visualize (via slices) this region of allowable (α, β, γ) triples, as well as the bounding region described by the Theorem 1.1 constraints. As suggested by Figure 6, these regions are geometrically rather interesting.

It is clear from such experiments that the constraints do a fairly good job bounding the allowable (α, β, γ) triples, but these experiments also clearly suggests that additional constraints, including linear constaints, should be added to the system to achieve a system that strongly bounds the triples. Also, by "uncommenting" a "define directive" in the program, it can be altered to include the "near constraints" mentioned in the previous note, but the improvement from these additional constraints is rather modest.

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Appendix

```
// tetrahedron_test.cpp (by M. Q. Rieck)
// Note: This is test code for the results in my "tetrahedron and toroids" paper.
// Note: This C++ program uses passing-by-reference. It can be easily converted to a C
// program by altering this aspect of function call, and by changing the includes.

#include <cstdio>
#include <cmath>

#define M 1000  // how many (alpha, beta, gamma) points (M^3)?
#define N 80    // how fine to subdivide the interval [0, pi]
#define O 1     // set higher to avoid low "tilt planes"
#define pi M_PI
// #define USE_NEAR_RULES

// The tau's are "tilt angles" for three planes, each containing one of the sidelines of
// the triangle ABC. Dihedral angle formulas are used to find the "view angles", alpha,
// beta and gamma, at the point of intersection of the three planes.
bool tilt_to_view_angles(double tau1, double tau2, double tau3, double cosA, double cosB,
  double cosC, double& alpha, double& beta, double& gamma, int& rejected) {
    double cos_tau1, cos_tau2, cos_tau3, sin_tau1, sin_tau2, sin_tau3;
    double cos_delta1, cos_delta2, cos_delta3, sin_delta1, sin_delta2, sin_delta3;
    cos_tau1 = cos(tau1), cos_tau2 = cos(tau2), cos_tau3 = cos(tau3);
    sin_tau1 = sin(tau1), sin_tau2 = sin(tau2), sin_tau3 = sin(tau3);
    cos_delta1 = sin_tau2 * sin_tau3 * cosA - cos_tau2 * cos_tau3;
    cos_delta2 = sin_tau3 * sin_tau1 * cosB - cos_tau3 * cos_tau1;
    cos_delta3 = sin_tau1 * sin_tau2 * cosC - cos_tau1 * cos_tau2;
    sin_delta1 = sqrt(1 - cos_delta1*cos_delta1);
```

```
    sin_delta2 = sqrt(1 - cos_delta2*cos_delta2);
    sin_delta3 = sqrt(1 - cos_delta3*cos_delta3);
    alpha = acos((cos_delta1 + cos_delta2 * cos_delta3) / (sin_delta2 * sin_delta3));
    beta  = acos((cos_delta2 + cos_delta3 * cos_delta1) / (sin_delta3 * sin_delta1));
    gamma = acos((cos_delta3 + cos_delta1 * cos_delta2) / (sin_delta1 * sin_delta2));
    if (alpha < 0 || alpha > pi || beta < 0 || beta > pi || gamma < 0 || gamma > pi ||
      alpha > beta+gamma || beta > gamma+alpha || gamma > alpha+beta || alpha+beta+
        gamma > 2*pi) { rejected++; return false; } else return true;
}

void clear_array(int a[N][N][N]) {
  for (int i=0; i<N; i++)
    for (int j=0; j<N; j++)
      for (int k=0; k<N; k++)
        a[i][j][k] = 0;
}

int ind(double angle) {
  int i = (int) (N*angle/pi);
  if (i < 0) i = 0;
  if (i >= N) i = N-1;
  return i;
}

void show_array(int a[N][N][N]) {
  printf("\n\n\n");
  for (int i=0; i<N; i++) {
    for (int j=0; j<N; j++) {
      for (int k=0; k<N; k++) {
        switch (a[i][j][k]) {
          case 0: printf("."); break;  // a "prohibited" cell that is empty
          case 1: printf("x"); break;  // a "prohibited" cell containing a data pt.
          case 2: printf(" "); break;  // an "allowable" cell that is empty
          case 3: printf("o");         // an "allowable" cell containing a data pt.
        }
      }
      printf("\n");
    }
    printf("\n");
    for (int k=0; k<N; k++) printf("_");
    printf("\n\n");
  }
  printf("\n");
}

int main() {

  int states[N][N][N], state, total, count0, count1, count2, count3, rejected = 0;
  double A, B, C, cosA, cosB, cosC, sinA, sinB, sinC, alpha, beta, gamma, tau1,
    tau2, tau3;

  // Set angles for an ACUTE base triangles ABC
  A =  8*pi/19; B =  6*pi/19; C =  5*pi/19;

  cosA = cos(A); cosB = cos(B); cosC = cos(C);
  sinA = sin(A); sinB = sin(B); sinC = sin(C);
  clear_array(states);

  // Use 3D array to record possible (alpha, beta, gamma) triples for given triangle
  for (int i=O; i<M-O; i++)
    for (int j=O; j<M-O; j++)
      for (int k=O; k<M-O; k++)
        if (tilt_to_view_angles(i*pi/M, j*pi/M, k*pi/M, cosA, cosB, cosC, alpha,
          beta, gamma, rejected)) states[ind(alpha)][ind(beta)][ind(gamma)] = 1;

  // Also use array to record which cells in the array are within system of bounds
  for (int i=0; i<N; i++)
    for (int j=0; j<N; j++)
```

```
      for (int k=0; k<N; k++) {
        alpha = (i+.5)*pi/N;
        beta  = (j+.5)*pi/N;
        gamma = (k+.5)*pi/N;
        if (
          A + beta + gamma  < 2*pi &&
          alpha + B + gamma < 2*pi &&
          alpha + beta + C  < 2*pi &&
          (alpha > A || beta  < B || beta  < C + alpha) &&
          (alpha > A || gamma < C || gamma < B + alpha) &&
          (beta  > B || gamma < C || gamma < A + beta ) &&
          (beta  > B || alpha < A || alpha < C + beta ) &&
          (gamma > C || alpha < A || alpha < B + gamma) &&
          (gamma > C || beta  < B || beta  < A + gamma) &&
          (alpha > A || cosC * cos(beta)  + cosB * cos(gamma) > 0) &&
          (beta  > B || cosA * cos(gamma) + cosC * cos(alpha) > 0) &&
          (gamma > C || cosB * cos(alpha) + cosA * cos(beta)  > 0)
#ifdef USE_NEAR_RULES
          && (alpha > A || B < C || B > A + C ||
            A * (beta + gamma - alpha) + (B - C) * (alpha + beta - gamma) < 2 * A * B)
          && (alpha > A || C < B || C > A + B ||
            A * (gamma + beta - alpha) + (C - B) * (alpha + gamma - beta) < 2 * A * C)
          && (beta  > B || C < A || C > B + A ||
            B * (gamma + alpha - beta) + (C - A) * (beta + gamma - alpha) < 2 * B * C)
          && (beta  > B || A < C || A > B + C ||
            B * (alpha + gamma - beta) + (A - C) * (beta + alpha - gamma) < 2 * B * A)
          && (gamma > C || A < B || A > C + B ||
            C * (alpha + beta - gamma) + (A - B) * (gamma + alpha - beta) < 2 * C * A)
          && (gamma > C || B < A || B > C + A ||
            C * (beta + alpha - gamma) + (B - A) * (gamma + beta - alpha) < 2 * C * B)
          && (alpha > A || B < C || beta  < B ||
            A * (beta + gamma - alpha) + (B - C) * (beta - alpha - gamma) < 2 * A * C)
          && (alpha > A || C < B || gamma < C ||
            A * (gamma + beta - alpha) + (C - B) * (gamma - alpha - beta) < 2 * A * B)
          && (beta  > B || C < A || gamma < C ||
            B * (gamma + alpha - beta) + (C - A) * (gamma - beta - alpha) < 2 * B * A)
          && (beta  > B || A < C || alpha < A ||
            B * (alpha + gamma - beta) + (A - C) * (alpha - beta - gamma) < 2 * B * C)
          && (gamma > C || A < B || alpha < A ||
            C * (alpha + beta - gamma) + (A - B) * (alpha - gamma - beta) < 2 * C * B)
          && (gamma > C || B < A || beta  < B ||
            C * (beta + alpha - gamma) + (B - A) * (beta - gamma - alpha) < 2 * C * A)
#endif
        ) states[i][j][k] += 2;
  }

  // Show slices of the array, indicating the nature of each cell.
  show_array(states);

  // Compute and display statistices for the given triangle ABC.
  total = count0 = count1 = count2 = count3 = 0;
  for (int i=0; i<N; i++)
    for (int j=0; j<N; j++)
      for (int k=0; k<N; k++) {
        switch (states[i][j][k]) {
          case 0: count0++; break;
          case 1: count1++; break;
          case 2: count2++; break;
          case 3: count3++;
        }
        total++;
  }
  printf("Number of   occupied   allowable cells:    %d\n", count3);
  printf("Number of unoccupied   allowable cells:    %d\n", count2);
  printf("Number of   occupied unallowable cells:    %d\n", count1);

  printf("Number of unoccupied unallowable cells:    %d\n", count0);
  printf("Total number of cells in the array:        %d\n", total);
```

```
  printf("Number of rejected calls for a data point: %d\n", rejected);
  printf("(Note: near the boundary, an \"unallowable\" cell might actually ");
  printf("have an allowable portion.)\n\n");

}
```

Declarations

Funding: None.
Conflicts of interest/Competing interests: None.
Availability of data and other material: Not applicable.
Code availability: The C++ program listed in the appendix is available at https://github.com/mqrieck/tetrahedron_test.cpp.
Authors' contribution: M. Q. Rieck is the sole author of this work.

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