UNDERSTANDING THE DELTOID PHENOMENON IN THE PERSPECTIVE 3-POINT (P3P) PROBLEM

Michael Q. Rieck
Mathematics and Computer Science Department, Drake University, Des Moines, IA 50311, USA
michael.rieck@drake.edu

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Abstract: Concerning the Perspective 3-Point (P3P) Problem, Grunert's system of three quadratic equations has a repeated solution if and only if the cubic polynomial introduced by Finsterwalder has a repeated root. This polynomial is here shown to be obtainable from a particularly simple cubic polynomial with complex coefficients via a simple Möbius transformation. This provides surprising geometric insight into the P3P problem. In particular, (1) the discriminant of Finsterwalder's polynomial can be written using the formula for the standard deltoid curve, and (2) this discriminant, when regarded as a function of camera position, vanishes on a surface that approaches a deltoid shape when the camera is moved infinitely far from the control points in a direction perpendicular to the control points plane (the "limit case"). These two facts have been previously reported, but obscure reasoning was required to establish them. In contrast, the present article uses the newly discovered cubic polynomial to easily produce the first fact, which then provides a basis for better understanding the second fact. Also presented are quartic polynomials whose real roots are the P3P solution point coordinates. A detailed geometric description of the P3P solution points in the "limit case" is also supplied.

1 Introduction

The classic problem known as the "Perspective 3-Point (P3P) Problem" has been reformulated and reexamined in a number of ways over the years. In its original form (Grunert, 1841), a pinhole camera's position and orientation are determined, though somewhat ambiguously, from the images in a photograph of three "control points" with known locations in space. To accomplish this, a system of three quadratic equations must be solved for the unknown distances from the camera's optical center to each of the control points.

As stated in (Haralick et al., 1994) (eqn. 1), this system of equations, henceforth referred to as "Grunert's system," is as follows:

$$\begin{cases} s_2^2 + s_3^2 - 2s_2s_3\cos\alpha = a^2 \\ s_3^2 + s_1^2 - 2s_3s_1\cos\beta = b^2 \\ s_1^2 + s_2^2 - 2s_1s_2\cos\gamma = c^2. \end{cases} \quad (1)$$

Here, a, b and c are the known distances between pairs of control points; α, β and γ are the angles between pairs of rays from the optical center to a control point. These angles are also presumed to be known, after computations involving the intrinsic properties of the pinhole camera. The s_j are the unknown distances that need to be determined by solving the system. Each of the three equations in the system is just an application of the Law of Cosines.

Unfortunately, the system typically produces multiple positive-valued solutions, as many as four of these, only one of which corresponds to the camera's actual position. Understanding how many such solutions result for given values of the parameters a, b, c, α, β and γ has been the focus of much investigation over the past few decades. For example, (Faugère et al., 2008), (Gao et al., 2003), (Rieck, 2015), (Rieck, 2018), (Rieck-Wang, 2021), (Wang et al., 2022). Similar to the situation for a single polynomial in a single variable, this question is closely related to determining which parameter values produce a repeated solution to the system, and this amounts to examining a quantity that is computed from these parameters.

Grunert's approach to solving the system Eq. (1)

leads to a quartic polynomial in a single variable. While a repeated root for this polynomial is required in order for the system to have a repeated solution, this is *not* sufficient. Indeed, the so-called "side-sharing" and "point-sharing" situations discussed in (Wang et al., 2020) describe the cases where Grunert's quartic polynomial has a repeated root but Grunert's system does not have a repeated solution.

However, the cubic polynomial introduced by S. Finsterwalder ((Haralick et al., 1994), eqn. 14) has a repeated root precisely when Grunert's system has a repeated solution. This is proved in Section 4 of the present paper. Prior to that, a connection is drawn, in Section 3, between Finsterwalder's cubic and a new cubic polynomial having complex coefficients. The discriminant of this latter cubic is easily expressed in terms of the formula for a well-known quartic curve called the "standard deltoid." It is a hypocycloid curve with three cusps. Aspects of this curve and its occurrence in "triangle geometry" are explored in Section 5. Then, in Section 6, an important technical fact is proved, concerning the non-trivial coefficients of the new cubic polynomial.

As explained in (Rieck, 2014), (Rieck, 2018), (Rieck-Wang, 2021), (Wang et al., 2022) and (Zhang-Hu, 2006), Grunert's system has a repeated positive solution if and only if one of the possible positions for the camera's optical center is on the circular cylinder through the circle containing the three control points. This is the so-called "danger cylinder." In this situation, any other points in space that correspond to other real solutions to the system must occur on a surface that has come to be called the "companion surface of the danger cylinder (*CSDC*)." Together with three special toroids, it separates space into regions corresponding to differing numbers of positive solutions to the system Eq. (1). (See (Rieck-Wang, 2021), Theorem 4.)

Following the practice adopted in (Rieck, 2015), (Rieck, 2018), (Rieck-Wang, 2021) and (Wang et al., 2022), a special Cartesian coordinate system will be used, though this in no way restricts the nature of the problem. The three (real) coordinates are called x, y and z. It will be helpful sometimes to let Z denote z^2. The coordinate system is chosen so that all three control points are on the unit circle in the xy-plane. In fact, the coordinates of these points are

$$(x_j, y_j, 0) = (\cos\phi_j, \sin\phi_j, 0) \quad (j = 1, 2, 3), \qquad (2)$$

with $\phi_1 + \phi_2 + \phi_3 = 0$. Such a coordinate system is always possible to obtain from a standard coordinate system, using an easily determined affine transformation. Details can be found in these other papers.

Table 1: Table of Symbols

Symbol	Meaning
x, y, z	Cartesian coordinates of a P3P solution point
x_j, y_j	Cartesian coordinates of a control point ($j = 1, 2, 3$)
a, b, c	distances between a pair of control points
d_1, d_2, d_3	alternative names for a, b, c
α, β, γ	viewing angles
$\theta_1, \theta_2, \theta_3$	alternative names for α, β, γ
s_j	distance between a P3P solution point and a control point ($j = 1, 2, 3$)
c_j	$\cos(\theta_j)$ ($j = 1, 2, 3$)
S_0	$1 - c_1c_2c_3$
S_j	$1 - c_j^2 = \sin^2(\theta_j)$ ($j = 1, 2, 3$)
T_j	$S_j - 2d_j^2 S_0/(d_1^2 + d_2^2 + d_3^2)$ ($j = 1, 2, 3$)
η^2	$T_1 + T_2 + T_3 =$ $1 - c_1^2 - c_2^2 - c_3^2 + 2c_1c_2c_3$
Z	z^2
ζ	$x + iy$ (a complex number)
ζ_j	$x_j + iy_j$ ($j = 1, 2, 3$)
(x_H, y_H)	$(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$, the orthocenter of the control points triangle
ζ_H	$x_H + iy_H = \zeta_1 + \zeta_2 + \zeta_3$
Δ_H	$(\zeta_2 - \zeta_1)(\zeta_3 - \zeta_1)(\zeta_3 - \zeta_2)$
G, H, I, J	coefficients of Finsterwalder's cubic polynomial
ζ_L	a complex number defined by P3P parameters only (see Eq. (21))
τ	indeterminate for inhomogeneous polynomial
τ_1, τ_2	indeterminates for homogeneous polynomial
$\sigma_1, \sigma_2, \sigma_3$	indeterminates for homogeneous polynomial
λ	σ_3/σ_1
ζ_0	an arbitrary complex number
ξ_j	a root of the cubic polynomial $\tau^3 - \zeta_0\tau^2 + \overline{\zeta_0}\tau - 1$ ($j = 1, 2, 3$)
ω_j	a square root of ξ_j ($j = 1, 2, 3$)
χ_0	$\omega_1 + \omega_2 + \omega_3$
χ_j	$2\omega_j - \chi_0$ ($j = 1, 2, 3$)
Δ	$(\xi_2 - \xi_1)(\xi_3 - \xi_1)(\xi_3 - \xi_2)$
Δ_L	Δ when ζ_0 is ζ_L
DC	the "danger cylinder" ($x^2 + y^2 = 1$, z free)
$CSDC$	the "companion surface" of DC

Also, following the practice introduced in (Rieck-Wang, 2021), it has proven to be extremely helpful to identify the xy-plane with the complex number plane by setting

$$\zeta = x + iy, \tag{3}$$

where $i^2 = -1$. In this way, the control points become identified with the unit complex numbers

$$\zeta_j = x_j + i y_j \ \ (j = 1,2,3). \tag{4}$$

Notice that $\overline{\zeta_j} = 1/\zeta_j$ $(j = 1,2,3)$ and that $\zeta_1\zeta_2\zeta_3 = 1$. Defining

$$\zeta_H = x_H + i y_H = \zeta_1 + \zeta_2 + \zeta_3, \tag{5}$$

it is straightforward to check that ζ_H is the orthocenter of the control points triangle, *i.e* the triangle having ζ_1, ζ_2 and ζ_3 as vertices. (The line containing ζ_H and ζ_1 is perpendicular to the line containing ζ_2 and ζ_3, and so forth.)

The *CSDC* surface is defined by a polynomial equation in x, y and Z. The highest degree of Z here is 4. See (Rieck-Wang, 2021), Lemma 4. When expressed as a polynomial equation in x, y and z, the (overall) degree is 12. See (Wang et al., 2022), Proposition 1. A curious fact concerning the *CSDC* surface is that when moving far from the control points, along a direction perpendicular to the plane containing the control points, the cross sections of this surface, cutting it with planes parallel to the control points plane, tend towards the standard deltoid curve. This claim, which was previously difficult to prove, is more easily established here, in Section 7, using the deltoidal nature of the discriminant.

Throughout this paper, ζ_1, ζ_2, ζ_3, ζ_H, a, b and c are assumed to be fixed. When discussing a "constant," it will be understood that this might depend on these numbers. However, other quantities, such as α, β, γ, s_1, s_2 and s_3, will sometimes be regarded as variables, varying as functions of the changing coordinates (ζ, z) of a moving camera's optical center. For the purposes of our discussion, a "constant" will never depend on these potentially varying quantities. Table 1 inventories most of the mathematical notation used in this document.

2 Main results

The cubic polynomial introduced by S. Finsterwalder is of concern in this paper. Its connection with the standard deltoid curve is captured in the first main result, as follows.

Theorem 1. *The discriminant of Finsterwalder's cubic polynomial Eq.* (28) *is equal to a nonzero constant times* Δ_L^2*, where*

$$\Delta_L^2 \;=\; \zeta_L^2\,\overline{\zeta_L^2} - 4\,(\zeta_L^3 + \overline{\zeta_L^3}) + 18\,\zeta_L\,\overline{\zeta_L} \;-\; 27, \tag{6}$$

and where ζ_L *is the complex number defined in Eq.* (21).

This is proved in Section 3.

Some of the importance of Finsterwalder's cubic in relation to Grunert's system can be seen in the next result.

Theorem 2. *Grunert's system (Eq.* (1)*) has a repeated solution if and only if Finsterwalder's cubic polynomial (Eq.* (28)*) has a repeated root, and so, if and only if its discriminant vanishes.*

This is proved in Section 4.

The quantity ζ_L (which depends on α, β and γ) can be expressed in a different manner, based on the Cartesian coordinates of any of the (real) P3P solution points, as follows.

Theorem 3.

$$\zeta_L \;=\; \zeta^2 \;-\; 2\overline{\zeta} + (\zeta\overline{\zeta} - 1)(\zeta^2 - \zeta_H\zeta - \overline{\zeta} + \overline{\zeta_H})\,/\,z^2, \tag{7}$$

where $\zeta = x + iy$*, and* (x,y,z) *are the coordinates of any (real) solution point for the P3P Problem (i.e a possible position in space of the camera's optical center).*

This is proved in Section 6.

The "limit case" discussed in Section 7, and in (Rieck, 2015), is particularly easy to analyze, and provides important insight into the general case. Here are some newly discovered facts concerning the (real) P3P solution points in the limit case. ("Real" here means that the coordinates x, y and z are real numbers.)

Theorem 4. *Consider the limit case, that is, the limiting situation as* $|z| \to \infty$*.*

1. *If* $\Delta_L^2 < 0$*, then the orthogonal projection of the (real) P3P solution points onto the xy-plane are the four points* χ_j *defined in Section 5 (upon setting* ζ_0 *to* ζ_L*). These points are inside the standard deltoid curve, but not on the unit circle. They form an orthocentric system whose orthic triangle has vertices that are the negative conjugates of the roots of the cubic polynomial Eq.* (23)*.*

2. *If instead $\Delta_L^2 > 0$, then the projections of the (real) P3P solution points are just the two χ_j for which $\chi_j^2 - 2\overline{\chi_j} = \zeta_L$. They are outside the standard deltoid curve.*

This is proved in Section 7.

3 A Special Cubic Polynomial

The method of Finsterwalder and the method of Grafarend-Lohse-Schaffrim (Haralick et al., 1994), as well as the method of Persson-Nordberg (Persson-Nordberg, 2018), all begin with essentially the same reasoning and produce essentially the same cubic polynomial. An equivalent approach will now be presented, one that leverages the symmetry of the system. This leads to a useful family of homogeneous cubic polynomials, one of which has a particularly simple, symmetric form, singling it out for special attention. To allow for easier symbolic manipulations, going forward, let us rename $\cos\alpha$, $\cos\beta$, $\cos\gamma$, a, b and c as c_1, c_2, c_3, d_1, d_2 and d_3, respectively.

Define three quadratic polynomials in s_1, s_2 and s_3 as follows:

$$\begin{aligned} p_1(s_1,s_2,s_3) &= s_2^2+s_3^2-2c_1s_2s_3-d_1^2, \\ p_2(s_1,s_2,s_3) &= s_3^2+s_1^2-2c_2s_3s_1-d_2^2, \\ p_3(s_1,s_2,s_3) &= s_1^2+s_2^2-2c_3s_1s_2-d_3^2. \end{aligned} \tag{8}$$

Solving Grunert's system means finding values for s_1, s_2 and s_3 that cause these three polynomials to simultaneously vanish. A general linear combination of these polynomials is the following:

$$\begin{aligned} p(\sigma_1,\sigma_2,\sigma_3;s_1,s_2,s_3) &= \sigma_1 p_1(s_1,s_2,s_3) \\ &+\sigma_2 p_2(s_1,s_2,s_3) + \sigma_3 p_3(s_1,s_2,s_3), \end{aligned} \tag{9}$$

for parameters σ_1, σ_2 and σ_3.

Of particular interest are the polynomials whose constant coefficient vanishes, that is,

$$d_1^2\sigma_1 + d_2^2\sigma_2 + d_3^2\sigma_3 = 0. \tag{10}$$

In this case, the equation $p(\sigma_1,\sigma_2,\sigma_3;s_1,s_2,s_3) = 0$ can be written in matrix form thus:

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} A_1 & B_3 & B_2 \\ B_3 & A_2 & B_1 \\ B_2 & B_1 & A_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 0, \tag{11}$$

where $A_1 = \sigma_2+\sigma_3$, $A_2 = \sigma_3+\sigma_1$, $A_3 = \sigma_1+\sigma_2$, and $B_j = -c_j\sigma_j$ $(j = 1,2,3)$.

All three of the methods mentioned earlier now essentially make the further assumption that the σ_j have been chosen so as to make the determinant of the 3×3 matrix vanish. This amounts to solving a cubic equation, as will be discussed next. In various ways, the three different methods then proceed to produce all of the positive solutions to Grunert's system.

The 3×3 determinant is a homogeneous cubic polynomial in the σ_j, specifically,

$$\begin{aligned} &(1-c_1^2)\sigma_1^2(\sigma_2+\sigma_3) + (1-c_2^2)\sigma_2^2(\sigma_3+\sigma_1) \\ &+(1-c_3^2)\sigma_3^2(\sigma_1+\sigma_2) + 2(1-c_1c_2c_3)\sigma_1\sigma_2\sigma_3. \end{aligned} \tag{12}$$

However, under the continuing assumption Eq. (10), we can reduce this to a homogeneous cubic polynomial in just two indeterminates (unknowns), in many different ways. Let us carefully examine two such approaches. Later on, we will see how these are related, and will observe that the indeterminates of either of the two homogeneous polynomials are merely linear transformations of the indeterminates of the other homogeneous polynomial.

One way to proceed is simply to use Eq. (10) to eliminate one of the σ_j from Eq. (12), and thereby obtain a homogeneous cubic polynomial in the other two σ_j.[1] For instance, if we eliminate σ_2 by setting it equal to $-(d_1^2\sigma_1 + d_3^2\sigma_3)/d_2^2$, we obtain d_2^{-4} times the following homogeneous version of Finsterwalder's cubic polynomial:

$$G\sigma_3^3 + H\sigma_1\sigma_3^2 + I\sigma_1^2\sigma_3 + J\sigma_1^3, \tag{13}$$

where

$$\begin{aligned} G &= d_3^2\,[d_3^2\,(1-c_2^2) - d_2^2\,(1-c_3^2)], \\ H &= d_2^2(d_2^2-d_1^2)\,(1-c_3^2) + d_3^2(d_3^2+2d_1^2)\,(1-c_2^2) \\ &\quad +2d_2^2d_3^2\,(c_1c_2c_3-1), \\ I &= d_2^2(d_2^2-d_3^2)\,(1-c_1^2) + d_1^2(d_1^2+2d_3^2)\,(1-c_2^2) \\ &\quad +2d_1^2d_2^2\,(c_1c_2c_3-1), \\ J &= d_1^2\,[d_1^2\,(1-c_2^2) - d_2^2\,(1-c_1^2)]. \end{aligned}$$

This agrees with (Haralick et al., 1994), eqn. 14, upon setting $\lambda = \sigma_3/\sigma_1$.

We now turn to a different method, one that employs substitution rather than elimination. Instead of eliminating one of the σ_i, we can introduce two new quantities (indeterminates), τ_1 and τ_2, and linearly relate these to the σ_i by setting

$$\sigma_i = \mu_{i1}\tau_1 + \mu_{i2}\tau_2 \;\; (i = 1,2,3), \tag{14}$$

[1] This resulting polynomial can then be solved by a classical method. Then, using the obtained values for the σ_j, the quadratic polynomial $p(\sigma_1,\sigma_2,\sigma_3;s_1,s_2,s_3)$ factors as a product of two linear factors in s_1, s_2 and s_3.

for constants μ_{ij}, subject to two restrictions:

$$d_1^2\mu_{1j}+d_2^2\mu_{2j}+d_3^2\mu_{3j} = 0 \quad (j=1,2). \tag{15}$$

To simplify the notation, let $S_j = 1-c_j^2$ $(j=1,2,3)$ and $S_0 = 1-c_1c_2c_3$. The determinant Eq. (12) can then be written thus:

$$\sum_{i=0}^{3}\sum_{j=0}^{3} \nu_{ij}S_j\tau_1^{3-i}\tau_2^i, \tag{16}$$

where

$$\begin{aligned}
\nu_{00} &= 2\mu_{11}\mu_{21}\mu_{31},\\
\nu_{01} &= \mu_{11}^2(\mu_{21}+\mu_{31}),\\
\nu_{02} &= \mu_{21}^2(\mu_{31}+\mu_{11}),\\
\nu_{03} &= \mu_{31}^2(\mu_{11}+\mu_{21}),\\
\nu_{10} &= 2(\mu_{12}\mu_{21}\mu_{31}+\mu_{11}\mu_{22}\mu_{31}+\mu_{11}\mu_{21}\mu_{32}),\\
\nu_{11} &= \mu_{11}^2(\mu_{22}+\mu_{32})+2\mu_{11}\mu_{12}(\mu_{21}+\mu_{31}),\\
\nu_{12} &= \mu_{21}^2(\mu_{32}+\mu_{12})+2\mu_{21}\mu_{22}(\mu_{31}+\mu_{11}),\\
\nu_{13} &= \mu_{31}^2(\mu_{12}+\mu_{22})+2\mu_{31}\mu_{32}(\mu_{11}+\mu_{21}),\\
\nu_{20} &= 2(\mu_{11}\mu_{22}\mu_{32}+\mu_{12}\mu_{21}\mu_{32}+\mu_{12}\mu_{22}\mu_{31}),\\
\nu_{21} &= \mu_{12}^2(\mu_{21}+\mu_{31})+2\mu_{11}\mu_{12}(\mu_{22}+\mu_{32}),\\
\nu_{22} &= \mu_{22}^2(\mu_{31}+\mu_{11})+2\mu_{21}\mu_{22}(\mu_{32}+\mu_{12}),\\
\nu_{23} &= \mu_{32}^2(\mu_{11}+\mu_{21})+2\mu_{31}\mu_{32}(\mu_{12}+\mu_{22}),\\
\nu_{30} &= 2\mu_{12}\mu_{22}\mu_{32},\\
\nu_{31} &= \mu_{12}^2(\mu_{22}+\mu_{32}),\\
\nu_{32} &= \mu_{22}^2(\mu_{32}+\mu_{12}),\\
\nu_{33} &= \mu_{32}^2(\mu_{12}+\mu_{22}).
\end{aligned} \tag{17}$$

At this stage, we introduce special choices for the values of the μ_{ij}, as follows:

$$\begin{aligned}
\mu_{11} &= 1/(\zeta_3-\zeta_2),\\
\mu_{21} &= 1/(\zeta_1-\zeta_3),\\
\mu_{31} &= 1/(\zeta_2-\zeta_1),\\
\mu_{12} &= \overline{\mu_{11}} = -\mu_{11}/\zeta_1,\\
\mu_{22} &= \overline{\mu_{21}} = -\mu_{21}/\zeta_2,\\
\mu_{32} &= \overline{\mu_{31}} = -\mu_{31}/\zeta_3.
\end{aligned} \tag{18}$$

These are reasonable choices for the parameters because of the next claim.

Lemma 1. *The choices for the μ_{ij} in Eq.* (18) *satisfy the two conditions Eq.* (15)*, and so are suitable for use in Eq.* (14) *so as to ensure Eq.* (10)*.*

Proof. $d_1^2 = (\zeta_2-\zeta_3)(\overline{\zeta_2}-\overline{\zeta_3})$, and similarly for d_2^2 and d_3^2. It is straightforward to check that $d_1^2\mu_{11}+d_2^2\mu_{21}+d_3^2\mu_{31} = 0$ and $d_1^2\mu_{12}+d_2^2\mu_{22}+d_3^2\mu_{32} = 0$. □

Define

$$\Delta_H = (\zeta_2-\zeta_1)(\zeta_3-\zeta_1)(\zeta_3-\zeta_2). \tag{19}$$

Lemma 2. *Δ_H is purely imaginary.*

Proof. $\overline{\Delta_H} = (1/\zeta_2-1/\zeta_1)(1/\zeta_3-1/\zeta_1)(1/\zeta_3-1/\zeta_2) = -\Delta_H/(\zeta_1\zeta_2\zeta_3)^2 = -\Delta_H$. □

Several other useful properties of the parameter choices are now easily checked.

Lemma 3. *The choices for the μ_{ij} in Eq.* (18) *satisfy the following equations:*

$$\begin{aligned}
&\mu_{11}\mu_{21}\mu_{31} = -1/\Delta_H\,,\ \mu_{12}\mu_{22}\mu_{32} = 1/\Delta_H\,,\\
&\mu_{11}^2(\mu_{21}+\mu_{31}) = \mu_{21}^2(\mu_{31}+\mu_{11}) =\\
&\mu_{31}^2(\mu_{11}+\mu_{21}) = 1/\Delta_H\,,\\
&\mu_{12}^2(\mu_{22}+\mu_{32}) = \mu_{22}^2(\mu_{32}+\mu_{12}) =\\
&\mu_{32}^2(\mu_{12}+\mu_{22}) = -1/\Delta_H\,.
\end{aligned}$$

Proof. The first two equations are immediate consequences of the definitions and Lemma 2. Now, $\mu_{21}+\mu_{31} = 1/(\zeta_1-\zeta_3)+1/(\zeta_2-\zeta_1) = (\zeta_2-\zeta_3)/[(\zeta_1-\zeta_3)(\zeta_2-\zeta_1)]$, and so, $\mu_{11}^2(\mu_{21}+\mu_{31}) = [1/(\zeta_3-\zeta_2)]^2(\zeta_2-\zeta_3)/[(\zeta_1-\zeta_3)(\zeta_2-\zeta_1)] = 1/\Delta_H$. Two similar equations follow by symmetry. The remaining equations to be proved follow by conjugation and by Lemma 2. □

An important quantity for analyzing the P3P Problem is the Gramian determinant associated with the unit vectors pointing from the camera's optical center to the control points. Following (Rieck, 2018), (Rieck-Wang, 2021) and (Wang et al., 2022), this quantity will be denoted by η^2, and can be computed thus:

$$\begin{aligned}
\eta^2 &= S_1+S_2+S_3-2S_0 =\\
&1-c_1^2-c_2^2-c_3^2+2c_1c_2c_3.
\end{aligned} \tag{20}$$

The following quantity, which was introduced in (Rieck-Wang, 2021), will also be needed here:

$$\begin{aligned}
\zeta_L = \eta^{-2}\,[\,&(\zeta_1^2+2\overline{\zeta_1})S_1+\\
&(\zeta_2^2+2\overline{\zeta_2})S_2+(\zeta_3^3+2\overline{\zeta_3})S_3-2\overline{\zeta_H}S_0\,].
\end{aligned} \tag{21}$$

We are now prepared to produce an interesting cubic polynomial that is closely related to Finsterwalder's cubic polynomial.

Lemma 4. *Using the values from Eq.* (17) *and Eq.* (18)*, the summation Eq.* (16) *reduces to*

$$(\eta^2/\Delta_H)\,[\tau_1^3-\zeta_L\tau_1^2\tau_2+\overline{\zeta_L}\tau_1\tau_2^2-\tau_2^3].$$

Proof. The coefficient of $S_1\tau_1^3$ in Eq. (16) is $\nu_{01} = \mu_{11}^2(\mu_{21}+\mu_{31}) = 1/\Delta_H$. Similarly for $S_2\tau_1^3$ and $S_3\tau_1^3$. The coefficient of $S_0\tau_1^3$ is $\nu_{00} = 2\mu_{11}\mu_{21}\mu_{31} = -2/\Delta_H$. Combining these, we see that the coefficient of τ_1^3 is η^2/Δ_H.

The coefficient of $S_1\tau_1^2\tau_2$ is $\nu_{11} = \mu_{11}^2(\mu_{22}+\mu_{32}) + 2\mu_{11}\mu_{12}(\mu_{21}+\mu_{31}) = (\mu_{11}/\mu_{12})^2\mu_{12}^2(\mu_{22}+\mu_{32}) + 2(\mu_{12}/\mu_{11})\mu_{11}^2(\mu_{21}+\mu_{31}) = \zeta_1^2(-1/\Delta_H) + 2(-\overline{\zeta_1})(1/\Delta_H) = -(\zeta_1^2+2\overline{\zeta_1})/\Delta_H$. Similarly for $S_2\tau_1^2\tau_2$ and $S_3\tau_1^2\tau_2$.

The coefficient of $S_0\tau_1^2\tau_2$ is $\nu_{10} = 2(\mu_{12}\mu_{21}\mu_{31} + \mu_{11}\mu_{22}\mu_{31} + \mu_{11}\mu_{21}\mu_{32}) = 2(\mu_{12}/\mu_{11} + \mu_{22}/\mu_{21} + \mu_{32}/\mu_{31})\mu_{11}\mu_{21}\mu_{31} = -2(\overline{\zeta_1}+\overline{\zeta_2}+\overline{\zeta_3})(-1/\Delta_H) = 2\overline{\zeta_H}/\Delta_H$. Combining, we see that the coefficient of $\tau_1^2\tau_2$ is $-\eta^2\zeta_L/\Delta_H$.

These facts, the preceding lemmas, and conjugation can now be used to quickly determine the other coefficients. Notice that ν_{0j} and ν_{3j} are conjugate, as are ν_{1j} and ν_{2j} $(j = 0,1,2,3)$. □

Because of its special form, the homogeneous cubic polynomial (in τ_1 and τ_2)

$$\tau_1^3 - \zeta_L\tau_1^2\tau_2 + \overline{\zeta_L}\tau_1\tau_2^2 - \tau_2^3 \tag{22}$$

has an associated inhomogeneous cubic polynomial (in τ)

$$\tau^3 - \zeta_L\tau^2 + \overline{\zeta_L}\tau - 1 \tag{23}$$

with a special form for its discriminant, specifically, Eq. (6). This results from the following general claim, whose proof is simply a matter of substituting specific coefficient values into the formula for the discriminant of a general cubic polynomial, and which is discussed in (MacKenzie, 1993).

Lemma 5. *Given a fixed complex number ζ_0, and an indeterminate τ, the discriminant of the cubic polynomial*

$$\tau^3 - \zeta_0\tau^2 + \overline{\zeta_0}\tau - 1 \tag{24}$$

is

$$\zeta_0^2\,\overline{\zeta_0^2} - 4(\zeta_0^3 + \overline{\zeta_0^3}) + 18\,\zeta_0\,\overline{\zeta_0} - 27. \tag{25}$$

Curiously, the equation for the standard deltoid curve, expressed in terms of a single complex variable ζ is

$$\zeta^2\,\overline{\zeta^2} - 4(\zeta^3 + \overline{\zeta^3}) + 18\,\zeta\,\overline{\zeta} - 27 = 0. \tag{26}$$

(See (Patterson, 1940), eqn. 3.3.) Actually, this curve has previously been studied as the discriminant of the cubic polynomial Eq. (24), with ζ in place of ζ_0. This and related ideas are explored in the next section.

One of the chief goals of this paper, namely proving Theorem 1, will now be accomplished via a series of lemmas. Towards this end, it will be helpful to introduce the following row vector:

$$M = \begin{bmatrix} 1/(\zeta_3-\zeta_2) & 1/(\zeta_1-\zeta_3) & 1/(\zeta_2-\zeta_1) \end{bmatrix}. \tag{27}$$

Lemma 6. *Let quantities q_1, q_2 and q_3 be such that $q_1+q_2+q_3 = 0$. Then,*

$$-\Delta_H \begin{bmatrix} q_1/d_1^2 & q_2/d_2^2 & q_3/d_3^2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} \zeta_1 & -\overline{\zeta_1} \\ \zeta_2 & -\overline{\zeta_2} \\ \zeta_3 & -\overline{\zeta_3} \end{bmatrix} \begin{bmatrix} M \\ \overline{M} \end{bmatrix}.$$

Proof. The first entry in the array on the right side of the equation to be established is $(q_1\zeta_1 + q_2\zeta_2 + q_3\zeta_3)/(\overline{\zeta_3}-\overline{\zeta_2}) - (q_1\overline{\zeta_1} + q_2\overline{\zeta_2} + q_3\overline{\zeta_3})/(\zeta_3 - \zeta_2)$, which equals $[\zeta_1/(\overline{\zeta_3}-\overline{\zeta_2}) - \overline{\zeta_1}/(\zeta_3-\zeta_2)]\,q_1 + [\zeta_2/(\overline{\zeta_3}-\overline{\zeta_2}) - \overline{\zeta_2}/(\zeta_3-\zeta_2)]\,q_2 + [\zeta_3/(\overline{\zeta_3}-\overline{\zeta_2}) - \overline{\zeta_3}/(\zeta_3-\zeta_2)]\,q_3$, which equals $[\zeta_1(\zeta_3-\zeta_2) - \overline{\zeta_1}(\overline{\zeta_3}-\overline{\zeta_2})]\,q_1/d_1^2 + [\zeta_2(\zeta_3-\zeta_2) - \overline{\zeta_2}(\overline{\zeta_3}-\overline{\zeta_2})]\,q_2/d_1^2 + [\zeta_3(\zeta_3-\zeta_2) - \overline{\zeta_3}(\overline{\zeta_3}-\overline{\zeta_2})]\,q_3/d_1^2$, which equals $[\zeta_1(\overline{\zeta_3}-\overline{\zeta_2}) - \overline{\zeta_1}(\zeta_3-\zeta_2) + \zeta_2\overline{\zeta_3} - \overline{\zeta_2}\zeta_3]\,q_1/d_1^2$, which equals $-\Delta_H\,q_1/d_1^2$. Similarly for the second and third entries in the array. □

Lemma 7. *Setting*

$$\begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix} \begin{bmatrix} 0 & -d_3^2/d_2^2 & 1 \\ 1 & -d_1^2/d_2^2 & 0 \end{bmatrix}$$

causes Eq. (12) *to become $1/d_2^4$ times Eq.* (13)*, but with σ_3 replaced with τ_1, and σ_1 replaced with τ_2. Moreover,*

$$\begin{bmatrix} 0 & -d_3^2/d_2^2 & 1 \\ 1 & -d_1^2/d_2^2 & 0 \end{bmatrix} = \frac{-1}{\Delta_H}\begin{bmatrix} d_3^2 & 0 \\ 0 & d_1^2 \end{bmatrix}.$$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \zeta_1 & -\overline{\zeta_1} \\ \zeta_2 & -\overline{\zeta_2} \\ \zeta_3 & -\overline{\zeta_3} \end{bmatrix} \begin{bmatrix} M \\ \overline{M} \end{bmatrix} = \frac{-1}{\Delta_H}\begin{bmatrix} d_3^2 & 0 \\ 0 & d_1^2 \end{bmatrix} \begin{bmatrix} \zeta_2-\zeta_3 & \overline{\zeta_3}-\overline{\zeta_2} \\ \zeta_2-\zeta_1 & \overline{\zeta_1}-\overline{\zeta_2} \end{bmatrix} \begin{bmatrix} M \\ \overline{M} \end{bmatrix}.$$

Proof. The first claim just amounts to the computation that results in Eq. (13), but with different variable names. The second claim is just a double application of Lemma 6. □

Lemma 8. *Consider the following two linear transformations of indeterminates:*

$$\begin{bmatrix} \tau_1' & \tau_2' \end{bmatrix} =$$
$$\frac{-1}{\Delta_H} \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix} \begin{bmatrix} d_3^2 & 0 \\ 0 & d_1^2 \end{bmatrix} \begin{bmatrix} \zeta_2 - \zeta_3 & \overline{\zeta_3} - \overline{\zeta_2} \\ \zeta_2 - \zeta_1 & \overline{\zeta_1} - \overline{\zeta_2} \end{bmatrix},$$
$$\begin{bmatrix} \sigma_1' & \sigma_2' & \sigma_3' \end{bmatrix} = \begin{bmatrix} \tau_1' & \tau_2' \end{bmatrix} \begin{bmatrix} M \\ \overline{M} \end{bmatrix}.$$

When σ_j' is used in place of σ_j $(j = 1,2,3)$ in Eq. (12)*, and expanded in terms of τ_1' and τ_2', the result is*

$$(\eta^2/\Delta_H)\,[\tau_1'^3 - \zeta_L \tau_1'^2 \tau_2' + \overline{\zeta_L} \tau_1' \tau_2'^2 - \tau_2'^3].$$

Then, when this is expanded in terms of τ_1 and τ_2, the result is

$$(1/d_2^4)\,[G\tau_1^3 + H\tau_1^2\tau_2 + I\tau_1\tau_2^2 + J\tau_2^3].$$

Proof. The first claim is just a restatement of Lemma 4, in matrix form, but using τ_1', τ_2', σ_1', σ_2' and σ_3' in place of τ_1, τ_2, σ_1, σ_2 and σ_3, respectively. Next, by Lemma 7,

$$\begin{bmatrix} \sigma_1' & \sigma_2' & \sigma_3' \end{bmatrix} =$$
$$\begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix} \begin{bmatrix} 0 & -d_3^2/d_2^2 & 1 \\ 1 & -d_1^2/d_2^2 & 0 \end{bmatrix},$$

and the second claim now follows too. □

We are now ready to establish one of the main results in this paper.

Proof of Theorem 1. Lemma 8 shows that a linear transformation of indeterminates exists that converts the homogeneous cubic polynomial Eq. (22) to the homogeneous cubic polynomial Eq. (13), and vice versa. Thus, a Möbius transformation of its indeterminate exists that converts the inhomogeneous cubic polynomial Eq. (23) to a nonzero constant times Finsterwalder's cubic polynomial,

$$G\lambda^3 + H\lambda^2 + I\lambda + J, \quad (28)$$

and vice-versa. Theorem 1 follows immediately from this fact and Lemma 5. □

4 Repeated P3P solutions

While the claim made in Theorem 2 seems to have been recognized by some, the author is unaware of any formal proof of it in the literature, apart from an appendix in (Rieck-Wang, 2021), which involves several tedious lemmas. A smoother and more direct proof will now be presented.

We will continue using the notation in Section 3, but will also use some of the notation in (Haralick et al., 1994) by setting $u = s_1/s_3$, $v = s_2/s_3$ and $\lambda = \sigma_3/\sigma_1$. Clearly, Eq. (9) implies the following:

$$A_1u^2 + A_2v^2 + A_3 + 2B_1v + 2B_2u + 2B_3uv = 0. \quad (29)$$

As was done previously, substitute $-(d_1^2\sigma_1 + d_3^2\sigma_3)\,/\,d_2^2$ and thus treat the coefficients of Eq. (29) as homogeneous linear functions of σ_1 and σ_3. Going a step further, let us dehomogenize this by setting $\sigma_1 = 1$, and so, $\sigma_3 = \lambda$, thereby obtaining a cubic polynomial in λ. In this way, we essentially obtain Finsterwalder's equation, (10) in (Haralick et al., 1994). This is a one-parameter family of quadratic equations in u and v, where λ is the parameter. The c_j and d_j here are presumed to be fixed. These equations correspond to a one-parameter family of conic sections in the uv-plane, though some are degenerate.

If fact, each degenerate conic section is a pair of lines, and corresponds to a value of λ that is one of the three roots of Eq. (12) using $\sigma_1 = 1$, $\sigma_3 = \lambda$ and $\sigma_2 = -(d_1^2 + d_3^2\lambda)/d_2^2$. In special cases, the "pair of lines" might actually be just a single line, but counted twice in the usual algebraic geometric sense. Such cases will be ignored here, but can be treated as limits of "generic" cases. Likewise, the rest of the discussion in this section will be generic.

Following Finsterwalder's method, four values (possibly complex) for the pair (u,v) are found that satisfy all of the equations in the family, corresponding to four points (possibly complex) on all of the conic sections. By Bézout's Theorem, there can be no other such points. Let (u_1,v_1), (u_2,v_2), (u_3,v_3) and (u_4,v_4) denote the four values of (u,v). We will speak of a "pairing" of the four points as a partitioning of them into two sets of size two points each. Note that there are three pairings. Each of the three degenerate conic section in the family of conic sections produces such a pairing by putting two of the points in the same set if they are on the same line of the degenerate conic section. The next result establishes a converse to this.

Lemma 9. *Given any pairing of the four points, there is a root to Eq.* (12) *(using $\sigma_1 = 1$, $\sigma_3 = \lambda$ and $\sigma_2 = -(d_1^2 + d_3^2\lambda)/d_2^2$) such that the corresponding degenerate conic section has any two of the four points on* *the same line if and only if they are in the same set of the given pairing.*

Proof. The general equation for a conic section in the plane involves six coefficients. However, scaling the equation by a constant factor does not change the curve it describes. Thus, there are five degrees of freedom in choosing a (possibly degenerate) conic section. But, if this curve is also required to pass through four specified points, then, this imposes four linear conditions on the coefficients. Working generically, we assume these to be linearly independent conditions. This reduces the number of degrees of freedom in the selection of a conic section to just one.

The situation in the Finsterwalder method is that we have a one-parameter family of conic sections that all pass through four particular points. By continuity and dimensional reasoning, there can be no other conic sections that pass through these four points. Now, consider any pairing of the four points. Each of the two sets in the pairing defines a line, and together these constitute a degenerate conic section. This conic section must be in the family of conic sections. Also, the degenerate conic section corresponds to one of the roots of the cubic equation.

□

We are now prepared to prove Theorem 2.

Proof of Theorem 2. From any value of (u,v), a value for (s_1^2,s_2^2,s_3^2) is uniquely determined via equations (4) and (5) in (Haralick et al., 1994). Moreover, the value of (s_1,s_2,s_3) can then be determined up to an overall factor of ± 1. The two resulting values of (s_1,s_2,s_3) can be practically regarded as "the same" solution to Grunert's system. Conversely, a value for (s_1,s_2,s_3) produces a unique value for (u,v).

Now, suppose that there is a repeat among the (u_j,v_j). Generically, there will be three distict (u_j,v_j), and, without loss of generality, we may assume that $(u_1,v_1)=(u_2,v_2)$, but that (u_1,v_1), (u_3,v_3) and (u_4,v_4) are distinct. A small continuous pertibation of the P3P paramters (the c_j and d_j) will slightly alter (continuously) the four ordered pairs, (u_j,v_j), and generically, the resulting ordered pairs, (u'_j,v'_j) $(j=1,2,3,4)$, will be distinct. But taking a pairing of these four ordered pairs, and the corresponding pairing of the four points in the *uv*-plane, and by also considering the pair of lines defined by this, we obtain a degenerate conic section in the family of conic sections, and so too, a corresponding root of the cubic polynomial in λ.

Here are the three possible pairings:

$$\begin{aligned}&\{\{(u'_1,v'_1),(u'_2,v'_2)\},\{(u'_3,v'_3),(u'_4,v'_4)\}\},\\&\{\{(u'_1,v'_1),(u'_3,v'_3)\},\{(u'_2,v'_2),(u'_4,v'_4)\}\},\\&\{\{(u'_1,v'_1),(u'_4,v'_4)\},\{(u'_2,v'_2),(u'_3,v'_3)\}\}.\end{aligned}$$

By reversing the continuous perturbation, and considering the effect on the pairings, there will result a repeated pairing (the second and third pairings). But, this implies a repeated root to the cubic polynomial in λ. Conversely, if the cubic polynomial has a repeated root, the Finsterwalder's method clearly leads to a repeated solution to Grunert's system.

□

5 Triangles and deltoids

Throughout this section, the focus is on constructions in the complex plane. $\Re(\tau)$ and $\Im(\tau)$ are used to denote the real and imaginary parts of a complex number τ, respectively. The investigation here is chiefly concerned with the cubic polynomial Eq. (24) and its discriminant Eq. (25).

Let ξ_1, ξ_2 and ξ_3 denote the three roots of the cubic polynomial Eq. (24). Clearly, $\xi_1+\xi_2+\xi_3=\zeta_0$, $\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2=\overline{\zeta_0}$, and $\xi_1\xi_2\xi_3=1$. Let

$$\Delta=(\xi_2-\xi_1)(\xi_3-\xi_1)(\xi_3-\xi_2), \tag{30}$$

and notice that Δ^2 equals the discriminant Eq. (25), and that this is real. The occurrence and study of polynomials of the form Eq. (24) can be found in earlier research efforts such as (Gongopadhyay et al., 2015), (MacKenzie, 1993) and (Patterson, 1940), as well as Chapter 19 of (Morley-Morley, 2014).

Two special values for ζ_0 are of importance in the study of P3P. First, when ζ_0 is taken to be ζ_H, we see that $\xi_j=\zeta_j$ (a control point; $j=1,2,3$). Second, when instead ζ_0 is taken to be ζ_L, as will be the case in subsequent sections of this document, insight can be gained into the nature of *CSDC* and the limit case of the P3P Problem.

From Theorem 4 of (MacKenzie, 1993), we can say that at least one of ξ_1, ξ_2 and ξ_3 must be on the unit circle, and that all three are on the unit circle if and only if the $\Delta^2\leq 0$. Geometrically, this condition means that ζ_0 is on or inside the standard deltoid given by the equation $\Delta^2 = 0$. In any case, without loss of generality, assume henceforth that ξ_1 is on the unit circle, *i.e.* $|\xi_1| = 1$.

When all three roots are on the unit circle, and are distinct, it is handy to regard them as the vertices of a triangle, and it is then straightforward to check that ζ_0 is the orthocenter of this triangle. Moreover, as indicated in Theorem 6 of (MacKenzie, 1993), $|\zeta_0|$ is less than, equal to, or greater than one depending on whether the triangle is acute, right or obtuse. For P3P, this is significant for the two special cases, when $\zeta_0 = \zeta_L$ and when $\zeta_0 = \zeta_H$.

A useful Möbius transformation in the present context is $\tau = (iw+1)/(iw-1)$. As explained in (MacKenzie, 1993), the polynomial Eq. (24) becomes $-i(iw-1)^{-3}$ times the following:

$$w^3 - \left(\frac{\Re(\zeta_0) - 3}{\Im(\zeta_0)}\right) w^2 + w - \left(\frac{\Re(\zeta_0)+1}{\Im(\zeta_0)}\right). \tag{31}$$

The coefficients of Eq. (31) are real numbers. Moreover, the Möbius transformation maps the real line (in the w-plane) to the unit circle (in the τ-plane). One can check that the discriminant of this polynomial is $-4\Delta^2/\Im(\zeta_0)^4$. (See page 243 of (MacKenzie, 1993).)

When $\Delta^2 < 0$, Eq. (31) has three real roots, corresponding to the three roots of Eq. (24) on the unit circle. However, when $\Delta^2 > 0$, Eq. (31) has only one real root. It also has two distinct complex roots that are complex conjugates of each other. Pulling these three roots back to the τ-plane, we find that the roots of Eq. (24) have the following form: $|\xi_1| = 1$ (by assumption), $\xi_2 = \xi r$ and $\xi_3 = \xi/r$ for some $r > 0$ and some ξ with $|\xi| = 1$. Since $r \neq 1$, by symmetry, we may assume that $r > 1$. Since $\xi_1\xi_2\xi_3 = 1$, we see that $\xi_1 = 1/\xi^2$.

A strong connection will be made in Section 7 between the solutions to the P3P Problem in the so-called "limit case," and upcoming notions in planar geometry discussed in the present section. To make this connection, some possibly new results in planar geometry will now be developed. Let ω_1, ω_2 and ω_3 be square roots of ξ_1, ξ_2 and ξ_3, respectively, chosen so that $\omega_1\omega_2\omega_3 = 1$. Let $\chi_0 = \omega_1 + \omega_2 + \omega_3$, and $\chi_j = 2\omega_j - \chi_0$ $(j = 1,2,3)$. As points in the complex plane, notice that ω_j is the midpoint of χ_0 and χ_j $(j = 1,2,3)$. Also, $-\omega_1$ is the midpoint of χ_2 and χ_3, and so forth. When $\Delta^2 > 0$, we may assume that $\omega_2 = \omega\rho$ and $\omega_3 = \omega/\rho$ for some ω and ρ with $|\omega| = 1$ and $\rho > 1$. Since $\omega_1\omega_2\omega_3 = 1$, $\omega_1 = 1/\omega^2$.

Lemma 10.

1. *When $\Delta^2 < 0$, the line containing χ_0 and χ_j intersects the line containing the other two χ's at the point $-\overline{\xi_j}$ $(j = 1,2,3)$.*
2. *When instead $\Delta^2 > 0$, and making the above assumptions, $-\overline{\xi_2}$ and $-\overline{\xi_3}$ are still the intersection points described in the $\Delta^2 < 0$ case, and $-\overline{\xi_1}$ still lies on the line through χ_0 and χ_1.*

Proof. First assume that $\Delta^2 < 0$. The line through χ_0 and χ_1 consists of points z for which $(\bar{z} - \overline{\chi_0})(z - \chi_1)$ is real, and this is so if and only if $\overline{\chi_0}\chi_1 - \overline{\chi_0}z - \chi_1\bar{z}$ is real. Observe that $i\Im[\overline{\chi_0}\chi_1] = i\Im[(\overline{\omega_1} + \overline{\omega_2} + \overline{\omega_3})(\omega_1 - \omega_2 - \omega_3)] = \omega_1\overline{\omega_2} - \overline{\omega_1}\omega_2 + \omega_1\overline{\omega_3} - \overline{\omega_1}\omega_3$. Setting $z = -\overline{\xi_1} = -\overline{\omega_1}^2$, we see that $i\Im[-\overline{\chi_0}z - \chi_1\bar{z}] = \overline{\omega_1}^2\overline{\omega_2} - \omega_1^2\omega_2 + \overline{\omega_1}^2\overline{\omega_3} - \omega_1^2\omega_3 = \overline{\omega_1}/\overline{\omega_3} - \omega_1/\omega_3 + \overline{\omega_1}/\overline{\omega_2} - \omega_1/\omega_2$. Since, $|\omega_2| = |\omega_3| = 1$, we find that $\Im[\overline{\chi_0}\chi_1 - \overline{\chi_0}z - \chi_1\bar{z}] = 0$, and so, $-\overline{\xi_1}$ is on the line containing χ_0 and χ_1.

Similarly, z is on the line through χ_2 and χ_3 if and only if $\overline{\chi_2}\chi_3 - \overline{\chi_2}z - \chi_3\bar{z}$ is real. Observe that $i\Im[\overline{\chi_2}\chi_3] = i\Im[(-\overline{\omega_1} + \overline{\omega_2} - \overline{\omega_3})(-\omega_1 - \omega_2 + \omega_3)] = \overline{\omega_1}\omega_2 - \omega_1\overline{\omega_2} + \omega_1\overline{\omega_3} - \overline{\omega_1}\omega_3$. Using $z = -\overline{\xi_1} = -\overline{\omega_1}^2$ again, we see that $i\Im[-\overline{\chi_2}z - \chi_3\bar{z}] = \overline{\omega_1}^2\overline{\omega_2} - \omega_1^2\omega_2 - \overline{\omega_1}^2\overline{\omega_3} + \omega_1^2\omega_3 = \overline{\omega_1}/\overline{\omega_3} - \omega_1/\omega_3 - \overline{\omega_1}/\overline{\omega_2} + \omega_1/\omega_2$. So again, since, $|\omega_2| = |\omega_3| = 1$, we find that $\Im[\overline{\chi_2}\chi_3 - \overline{\chi_2}z - \chi_3\bar{z}] = 0$, and so, $-\overline{\xi_1}$ is on the line containing χ_2 and χ_3. Symmetrical reasoning completes the $\Delta^2 < 0$ case.

Now assume that $\Delta^2 > 0$ instead. Here we have $|\omega_1| = 1$, but $\overline{\omega_2} = 1/\omega_3$ and $\overline{\omega_3} = 1/\omega_2$. Rechecking the earlier computations it is quickly seen that $-\overline{\xi_1}$ is still on the line containing χ_0 and χ_1. The claims concerning $-\overline{\xi_2}$ and $-\overline{\xi_3}$ can also be readily checked. □

Figure 1 shows an example for the $\Delta^2 < 0$ case. The deltoid curve and unit circle are drawn here. The diamond (tilted square) shows the position of the point ζ_L. The solid circular dots mark the points ξ_1, ξ_2, ξ_3. The small squares mark the points $\pm\omega_1, \pm\omega_2, \pm\omega_3$. The small triangles mark the points $\chi_0, \chi_1, \chi_2, \chi_3$. Finally, the open circular dots mark the points $-\overline{\xi_1}, -\overline{\xi_2}, -\overline{\xi_3}$. Dotted lines show the collinearity of various points.

Lemma 11. *If $\Delta^2 < 0$, then χ_0 is the orthocenter of the triangle with vertices ω_1, ω_2, ω_3; χ_1 is the orthocenter of the triangle with vertices ω_1, $-\omega_2$, $-\omega_3$; χ_2 is the orthocenter of the triangle with vertices $-\omega_1$,*

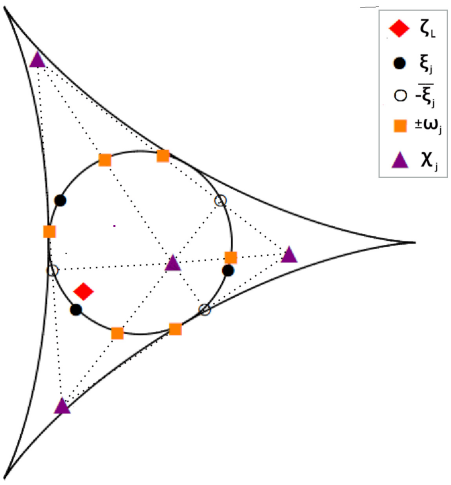

Figure 1: The $\Delta^2 < 0$ case

ω_2, $-\omega_3$; and χ_3 is the orthocenter of the triangle with vertices $-\omega_1$, $-\omega_2$, ω_3.

Proof. The line through ω_1 and χ_0 is parallel to the line through the origin and the point $\omega_2+\omega_3$. The line through ω_2 and ω_3 is parallel to the line through the origin and the point $\omega_2-\omega_3$. The first two lines are perpendicular to the second two lines if and only if $(\omega_2+\omega_3)/(\omega_2-\omega_3)$ is purely imaginary. This is so if and only if $(\omega_2+\omega_3)(\overline{\omega_2}-\overline{\omega_3})$ is purely imaginary. But this quantity equals $\overline{\omega_2}\omega_3-\omega_2\overline{\omega_3}$, which is purely imaginary. By symmetry, we see that χ_0 is the orthocenter of the triangle whose vertices are ω_1, ω_2 and ω_3. Similar reasoning establishes the other claims. □

Lemma 12. *If $\Delta^2 < 0$, then the set of points $\{\chi_0,\chi_1,\chi_2,\chi_3\}$ is an orthocentric system, and the points $-\overline{\xi_1}$, $-\overline{\xi_2}$ and $-\overline{\xi_3}$ are the vertices of its orthic triangle.*

Proof. Reasoning as in the proof of Lemma 11, examine $(\chi_0-\chi_1)(\overline{\chi_2}-\overline{\chi_3})$. This equals $4(\omega_2+\omega_3)(\overline{\omega_2}-\overline{\omega_3})$, which is purely imaginary. Hence the line through χ_0 and χ_1 is perpendicular to the line through χ_2 and χ_3. By symmetry, the four points χ_j $(j=0,1,2,3)$ form an orthocentric system. By Lemma 10, the points $-\overline{\xi_j}$ $(j=1,2,3)$ are the vertices of the orthocentric system's orthic triangle. □

Lemma 13. *If $\Delta^2 < 0$, then $\chi_j^2-2\overline{\chi_j}=\zeta_0$ $(j=0,1,2,3)$.*

Proof. $\chi_0^2-2\overline{\chi_0}=(\omega_1+\omega_2+\omega_3)^2-2(1/\omega_1+1/\omega_2+1/\omega_3)=(\omega_1+\omega_2+\omega_3)^2-2(\omega_2\omega_3+\omega_3\omega_1+\omega_1\omega_2)=\xi_1+\xi_2+\xi_3=\zeta_0$. Similar reasoning establishes the other claims. □

Figure 2 is similar to Figure 1, but is for the $\Delta^2 > 0$ case. Here $-\overline{\xi_1}$ (the small open circle on the left) now lies on the line containing χ_0 and χ_1 (the triangles on the left), but not on the line containing χ_2 and χ_3 (the triangles on the right). On the other hand, $-\overline{\xi_2}$ and $-\overline{\xi_3}$ (the other two small open circles) are again each the intersection point of two lines, each line containing a pair of the χ_j's. The dashed line demonstrates the next result.

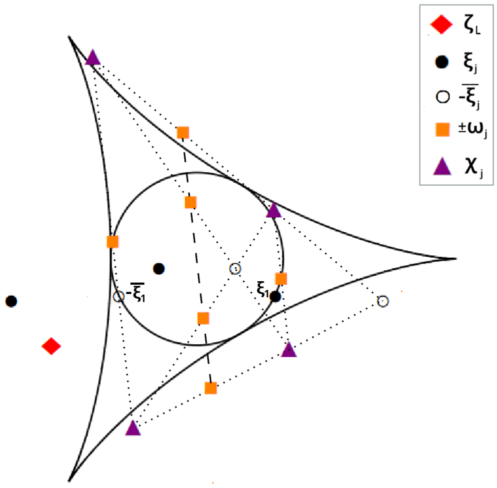

Figure 2: The $\Delta^2 > 0$ case

Lemma 14. *If $\Delta^2 > 0$ and $|\xi_1| = 1$, then $0, \omega_2$, ω_3, $-\omega_2$ and $-\omega_3$ are collinear. Moreover, the line containing these points is parallel to the line containing χ_0, χ_1 and ω_1, as well as the line containing χ_2, χ_3 and $-\omega_1$.*

Proof. As previously indicated, we may assume that $\omega_1 = 1/\omega^2, \omega_2 = \omega\rho$ and $\omega_3 = \omega/\rho$ for some $\rho > 1$ and some ω with $|\omega| = 1$. The points $0, \omega_2$, ω_3, $-\omega_2$ and $-\omega_3$ are clearly collinear, and their common line ℓ is in the direction $\pm\omega$.

The line containing χ_0 and χ_1 is parallel to the line through the origin and the point $\chi_0-\chi_1 = 2(\omega_2+\omega_3) = 2\omega(\rho+1/\rho)$, which is the line ℓ. The line containing χ_2 and χ_3 is parallel to the line through the origin and the point $\chi_2-\chi_3 = 2(\omega_2-\omega_3) = 2\omega(\rho-1/\rho)$, which is also the line ℓ. □

Lemma 15. *If $\Delta^2 > 0$ and $|\xi_1| = 1$, then $\chi_0^2 - 2\overline{\chi_0} = \chi_1^2 - 2\overline{\chi_1} = \zeta_0$, but $\chi_2^2 - 2\overline{\chi_2} \neq \zeta_0$ and $\chi_3^2 - 2\overline{\chi_3} \neq \zeta_0$. However, $[\chi_2^2 - 2\overline{\chi_2} + \chi_3^2 - 2\overline{\chi_3}]/2 = \zeta_0$.*

Proof. Continue the assumptions at the start of the proof of Lemma 14. $\chi_0^2 - 2\overline{\chi_0} = (1/\omega^2 + \omega\rho + \omega/\rho)^2 - 2[\omega^2 + \rho/\omega + 1/(\omega\rho)] = 1/\omega^4 + \omega^2\rho^2 + \omega^2/\rho^2 + 2\rho/\omega + 2/(\omega\rho) + 2\omega^2 - 2\omega^2 - 2\rho/\omega - 2/(\omega\rho) = 1/\omega^4 + \omega^2\rho^2 + \omega^2/\rho^2 = \xi_1 + \xi_2 + \xi_3 = \zeta_0$. A quick inspection shows that we get the same result if we begin with χ_1 in place of χ_0.

$\chi_2^2 - 2\overline{\chi_2} = (-1/\omega^2 + \omega\rho - \omega/\rho)^2 - 2[-\omega^2 + \rho/\omega - 1/(\omega\rho)] = 1/\omega^4 + \omega^2\rho^2 + \omega^2/\rho^2 - 2\rho/\omega + 2/(\omega\rho) - 2\omega^2 + 2\omega^2 - 2\rho/\omega + 2/(\omega\rho) = 1/\omega^4 + \omega^2\rho^2 + \omega^2/\rho^2 - 4\rho/\omega + 4/(\omega\rho)$, which does not equal ζ_0, since $\rho^2 \neq 1$. Likewise, $\chi_3^2 - 2\overline{\chi_3} = 1/\omega^4 + \omega^2\rho^2 + \omega^2/\rho^2 + 4\rho/\omega - 4/(\omega\rho)$, from which the rest follows. □

6 The quantity ζ_L

The quantity ζ_L defined in Eq. (21) first appeared in (Rieck-Wang, 2021), where it was proved to equal a quantity that depends explicitly, and in a simple way, on the coordinates ζ $(= x + iy)$ and z of the camera's optical center. (See Theorem 1 in (Rieck-Wang, 2021).) In the present paper, this is Theorem 3. The goal now is to prove this claim in a manner that is more direct and revealing than the approach taken in (Rieck-Wang, 2021).

It will be helpful to define

$$T_j = S_j - 2\,d_j^2\,S_0\,/\,(d_1^2 + d_2^2 + d_3^2) \quad (j = 1,2,3). \quad (32)$$

The polynomials in c_1, c_2 and c_3 described in Section 5 of (Rieck, 2018) are just the linear combinations T_1, T_2 and T_3 with constant coefficients (that can depend on the constants ζ_1, ζ_2 and ζ_3). These are the polynomials of the form (4) (there) that satisfy (5) (there), which have special properties. Notice that $\eta^2 = T_1 + T_2 + T_3$. Other quantities of interest, seen in (Rieck, 2014), (Rieck, 2015) and (Rieck, 2018), are $d_3^2 S_2 - d_2^2 S_3$, $d_1^2 S_3 - d_3^2 S_1$ and $d_2^2 S_1 - d_1^2 S_2$.

Lemma 16.

$$\frac{d_3^2 S_2 - d_2^2 S_3}{\eta^2} = \frac{d_3^2 T_2 - d_2^2 T_3}{\eta^2} = \frac{1 - \zeta_1\,\zeta_L + \zeta_1^2\,\overline{\zeta_L} - \zeta_1^3}{\zeta_1^2\,(\zeta_2 - \zeta_3)},$$

and similar equations obtained by cycling the indices.

Proof. It is straightforward to check that $d_3^2 S_2 - d_2^2 S_3 = d_3^2 T_2 - d_2^2 T_3$. Now, $\zeta_1^2(\zeta_2 - \zeta_3)(d_3^2 T_2 - d_2^2 T_3) = \zeta_1^2(\zeta_2 - \zeta_3)[(\zeta_2 - \zeta_1)(\overline{\zeta_2} - \overline{\zeta_1})\,T_2 - (\zeta_3 - \zeta_1)(\overline{\zeta_3} - \overline{\zeta_1})\,T_3] = [(\zeta_3 - \zeta_2)(1 - \zeta_2^2\zeta_3)^2/(\zeta_2^4\zeta_3^3)]\,T_2 + [(\zeta_2 - \zeta_3)(1 - \zeta_2\zeta_3^2)^2/(\zeta_2^3\zeta_3^4)]\,T_3$. Also, $\eta^2\,(1 - \zeta_1\,\zeta_L + \zeta_1^2\,\overline{\zeta_L} - \zeta_1^3) = (1 - \zeta_1\overline{\zeta_1})^2\,T_1 + [1 - \zeta_1(\zeta_2^2 + 2\overline{\zeta_2}) + \zeta_1^2(\overline{\zeta_2}^2 + 2\zeta_2) - \zeta_1^3]\,T_2 + [1 - \zeta_1(\zeta_3^2 + 2\overline{\zeta_3}) + \zeta_1^2(\overline{\zeta_3}^2 + 2\zeta_3) - \zeta_1^3]\,T_3$, which also equals $[(\zeta_3 - \zeta_2)(1 - \zeta_2^2\zeta_3)^2/(\zeta_2^4\zeta_3^3)]\,T_2 + [(\zeta_2 - \zeta_3)(1 - \zeta_2\zeta_3^2)^2/(\zeta_2^3\zeta_3^4)]\,T_3$. □

Lemma 17.

$$\frac{d_3^2 S_2 - d_2^2 S_3}{\eta^2} = \frac{\zeta_1}{\zeta_2 - \zeta_3}\Big\{\,[\,\overline{\zeta_1}^{\,3} - \overline{\zeta_1}^{\,2}\,(\zeta^2 - 2\overline{\zeta})$$
$$+\,\overline{\zeta_1}\,(\overline{\zeta}^2 - 2\zeta) - 1] + [\tfrac{1 - \zeta\overline{\zeta}}{z^2}]\,\cdot$$
$$[\,\overline{\zeta_1}^{\,2}\,(\zeta^2 - \zeta_H\,\zeta - \overline{\zeta} + \overline{\zeta_H})$$
$$-\,\overline{\zeta_1}\,(\overline{\zeta}^2 - \overline{\zeta_H}\,\overline{\zeta} - \zeta + \zeta_H)\,]\,\Big\}$$

and similar equations obtained by cycling the indices.

Proof. Using $c_1 = (s_2^2 + s_3^2 - d_1^2)/(2 s_2 s_3)$, etc., we find that $S_1 = 1 - c_1^2 = (2 s_2^2 s_3^2 + 2 d_1^2 s_2^2 + 2 d_1^2 s_3^2 - d_1^4 - s_2^4 - s_3^4)\,/\,(4 s_2^2 s_3^2)$. By considering the volume of a tetrahedron, in a couple ways, it can be seen that $\eta^2 = (d_1^2 d_2^2 d_3^2 z^2)\,/\,(4 s_1^2 s_2^2 s_3^2)$. This is proved in Lemma 2 of (Rieck, 2014). Therefore, $(d_3^2 S_2 - d_2^2 S_3)\,/\,\eta^2 = [2(d_3^2 - d_2^2) s_1^2 s_2^2 s_3^2 - d_3^2 s_1^4 s_2^2 + d_2^2 s_1^4 s_3^2 + d_3^2 s_2^4 s_3^2 - d_2^2 s_2^2 s_3^4 + 2 d_2^2 d_3^2 s_1^2 (s_2^2 - s_3^2) - d_2^4 d_3^2 s_2^2 + d_2^2 d_3^4 s_3^2]\,/\,[d_1^2 d_2^2 d_3^2 z^2]$. Substituting using $s_1^2 = (\zeta - \zeta_1)(\overline{\zeta} - \overline{\zeta_1}) + z^2$, etc., $d_1^2 = (\zeta_2 - \zeta_3)(\overline{\zeta_2} - \overline{\zeta_3})$, etc., and $\overline{\zeta_j} = 1\,/\,\zeta_j\;(j = 1,2,3)$ establishes the formula in the lemma. □

Proof of Theorem 3. Define two cubic polynomials, in a complex variable τ, as follows:

$$\begin{aligned} p(\tau) &= \tau^3 - \zeta_L \tau^2 + \overline{\zeta_L}\tau - 1 \\ q(\tau) &= \tau^3 - \zeta'_L \tau^2 + \overline{\zeta'_L}\tau - 1, \end{aligned}$$

where ζ'_L is the right side of equation Eq. (7). $p(\tau)$ is just Eq. (23). The goal now is to prove that $p(\tau)$ and $q(\tau)$ are really the same polynomial, and therefore that $\zeta_L = \zeta'_L$, which is the claim made in Theorem 3.

By Lemma 16 and Lemma 17, $p(\overline{\zeta_1}) = q(\overline{\zeta_1})$, and by symmetry, $p(\overline{\zeta_2}) = q(\overline{\zeta_2})$ and $p(\overline{\zeta_3}) = q(\overline{\zeta_3})$. Clearly, $p(0) = q(0)$. Since two cubic polynomials agree at four values, they are actually the same polynomial. This establishes Theorem 3.

□

7 P3P solutions in the limit

We will now focus on the so-called “limit case,” studied in (Rieck, 2015). The idea is to consider what happens as the Cartesian coordinate z of the camera’s optical center (a P3P solution point) becomes very large in relation to $|\zeta_H|$ and $|\zeta|$. In this case, the right side of Eq. (7) tends to $\zeta^2 - 2\overline{\zeta}$. We may imagine a collection of “limit points,” where $z = \infty$, for which the right side of Eq. (7) equals $\zeta^2 - 2\overline{\zeta}$.

Formally, this means partially compactifying real space in a manner that is not one-point compactification, nor is it the compactification that results in projective space. Instead, we are simply compactifying each “vertical line” (constant-ζ line) by adding a point at infinity. The plane consising of all of the points at infinity constitute what is meant by the “limit case” here.

The discriminant Δ_L^2 given by Eq. (6), equals zero on the “danger cylinder,” which is the circular cylinder given by $\zeta\overline{\zeta} = 1$. Δ_L^2 also tends to zero on the (non-circular) cylinder given by the formula for the standard deltoid Eq. (26), as z tends to infinity. This is easily understood, because in the limit as $z \to \infty$, we have $\zeta_L = \zeta^2 - 2\overline{\zeta}$, and because of the following fact.

Lemma 18. *The mapping $\zeta \mapsto \zeta^2 - 2\overline{\zeta}$, in the complex plane, takes points on the unit circle and points on the standard deltoid to points on the standard deltoid. No other points are mapped to the standard deltoid.*

Proof. Substitute $\zeta^2 - 2\overline{\zeta}$ for ζ_0 in Eq. (25) to get $(\zeta^2 - 2\overline{\zeta})^2(\overline{\zeta}^2 - 2\zeta)^2 - 4[(\zeta^2 - 2\overline{\zeta})^3 + (\overline{\zeta}^2 - 2\zeta)^3] + 18(\zeta^2 - 2\overline{\zeta})(\overline{\zeta}^2 - 2\zeta) - 27 = \zeta^4\overline{\zeta}^4 - 4\zeta_5\overline{\zeta} - 4\zeta\overline{\zeta}^5 + 16\zeta^3\overline{\zeta}^3 + 8\zeta^4\overline{\zeta} + 8\zeta\overline{\zeta}^4 - 62\zeta^2\overline{\zeta}^2 - 4\zeta^3 - 4\overline{\zeta}^3 + 72\zeta\overline{\zeta} - 27 = (\zeta^2\overline{\zeta}^2 - 2\zeta\overline{\zeta} + 1)(\zeta^2\overline{\zeta^2} - 4(\zeta^3 + \overline{\zeta^3}) + 18\zeta\overline{\zeta} - 27) = (\zeta\overline{\zeta} - 1)^2[\zeta^2\overline{\zeta^2} - 4(\zeta^3 + \overline{\zeta^3}) + 18\zeta\overline{\zeta} - 27]$. The claim is now evident. □

This proof yields an equation that seems to “magically appear.” In fact, it occurs “naturally” when one studies simple operations on the sort of triangles considered in Section 5. Such ideas are explored in (Rieck, 2024).

We see that the points on the *CSDC* (with $|z| < \infty$), discussed in Section 1, are such that their ζ values are arbitrarily close to the standard deltoid, provided that their $|z|$ is sufficiently large. This explains the “deltoid phenomenon” that was first observed in (Rieck, 2015).

In the limit case, when $\Delta_L^2 < 0$ there are four P3P solution points. But when $\Delta_L^2 > 0$ there are only two P3P solution points. This was established in Corollary 1 of (Rieck, 2015) (though the sign used for Δ_L^2 there was the opposite of the sign used here). Theorem 4 of the current paper is concerned with these points. Here now is a proof of it.

Proof of Theorem 4. $\zeta_L = \zeta^2 - 2\overline{\zeta}$ for a P3P solution point, in the limit case. Now, for given ζ_L, there are at most four solutions to $\zeta_L = \zeta^2 - 2\overline{\zeta}$. To see this, notice that this equation implies that $\overline{\zeta_L} = \overline{\zeta}^2 - 2\zeta$. These together imply that $\overline{\zeta_L} = (\zeta^2 - \zeta_L)^2/4 - 2\zeta$, and so, $\zeta^4 - 2\zeta_L\zeta^2 - 8\zeta + \zeta_L^2 - 4\overline{\zeta_L} = 0$. This is a quartic equation in ζ, and so it has no more than four solutions.

Lemma 18 makes it clear, in the limit case, that when $\Delta_L^2 = 0$, the ζ coordinate of the point is on the union of the unit circle and the standard deltoid. Also, the occurrence of $\zeta\overline{\zeta} - 1$ raised to the second power in the proof of Lemma 18 means that the sign of Δ_L^2 does not change when the optical center of the camera crosses the danger cylinder (in the limit case). It is then straightforward to check that the ζ coordinate is outside the standard deltoid when $\Delta_L^2 > 0$, but is inside the standard deltoid, yet not on the unit circle, when $\Delta_L^2 < 0$.

In the $\Delta_L^2 < 0$ case, Lemma 13 says that the four χ_j are solutions to $\zeta_L = \zeta^2 - 2\overline{\zeta}$, and so there can be no other solution. By Corollary 1 in (Rieck, 2015), there are exactly four (real) P3P solution points when $\Delta_L^2 < 0$, so the χ_j must be the orthogonal projection onto the xy-plane of these four points. (The sign convention used for Δ_L^2 there is the opposite of the one used here.)

Corollary 1 in (Rieck, 2015) also indicates that when $\Delta_L^2 > 0$, there are exactly two (real) P3P solution points. To establish that their orthogonal projections onto the xy-plane are χ_0 and χ_1, first observe that Lemma 15 indicates that these satisfy the $\zeta_L = \zeta^2 - 2\overline{\zeta}$ requirement (with $\zeta_0 = \zeta_L$). It then suffices to show that the equation $\zeta_L = \zeta^2 - 2\overline{\zeta}$ has only two solutions. Using $\zeta_L = -1 - \mathcal{R} + i\mathcal{L}$, with $\mathcal{L}$ and $\mathcal{R}$ real, and as defined in (Rieck, 2018) and (Rieck-Wang, 2021), we are essentially seeking real-valued solutions to the following system, solving for the unknowns x and y:

$$\begin{cases} \mathcal{R} &= y^2-(x-1)^2 \\ \mathcal{L} &= 2(x+1)y \end{cases}$$

Thus we are asking about the intersection points for a pair of rectangular hyperbolas, as discussed in (Rieck, 2015) and in (Rieck, 2018). In fact, the essence of Corollary 1 in (Rieck, 2015) is that if this system has a (real) solution, then it has exactly two such solutions if $\Delta_L^2 > 0$, and exactly four solutions if $\Delta_L^2 < 0$. Lemma 20 of (Rieck, 2018) assures that there is always at least one (real) solution. □

Note: In Figure 2, where $\Delta_L^2 > 0$, only the two small triangles furthest to the left (χ_0 and χ_1) mark the orthogonal projections of P3P solution points onto the xy-plane (in the limit case).

8 The circumcubic, circumquartics and coordinate quartics

The rectangular coordinates of the camera's optical center continue to be denoted (x,y,z), and we take $\zeta = x+iy$ and $Z = z^2$. The quantities $\mathcal{L}$ and $\mathcal{R}$, used in the proof of Theorem 4, are needed here as well. Again, $\zeta_L = -1-\mathcal{R}+i\mathcal{L}$. Also needed are these quantities from Sections 1, 2 and 3: x_1, x_2, x_3, y_1, y_2, y_3, x_H, y_H, $c_1 = \cos\alpha$, $c_2 = \cos\beta$, $c_3 = \cos\gamma$, $S_0 = 1-c_1c_2c_3$, $S_1 = 1-c_1^2$, $S_2 = 1-c_2^2$, $S_3 = 1-c_3^2$.

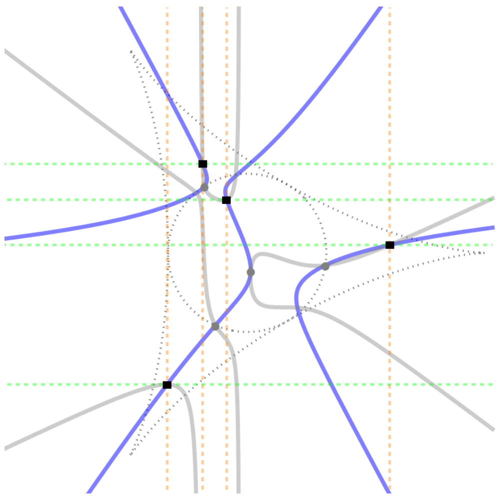

Figure 3: Circumcubic and circumquartic curves

When Eq. (7) from Theorem 3 is expanded into its real and imaginary parts, these formulas (first appearing in (Rieck, 2018)) result:

$$\mathcal{L} = \left[\, (x^2+y^2-1)(y+2xy-x_Hy-y_Hx-y_H) + 2\,(1+x)\,yZ\,\right] / \, Z$$

and

$$\mathcal{R} = \left[\, (x^2+y^2-1)(x-x^2+y^2+x_Hx-y_Hy-x_H) - (x-y-1)(x+y-1)Z\,\right] / Z.$$

When Z is eliminated from these two equations, there results the following cubic polynomial in x and y:

$$\begin{aligned} &y_Hx^3 + (3-x_H)x^2y + y_Hxy^2 - (1+x_H)y^3 \\ &-(\mathcal{L}+y_H)x^2 - 2(1+\mathcal{R}+x_H)xy + (\mathcal{L}+y_H)y^2 \\ &+(\mathcal{L}+x_H\mathcal{L}+y_H\mathcal{R}-y_H)x + (3x_H+x_H\mathcal{R} \\ &-y_H\mathcal{L}-\mathcal{R}-1)y - x_H\mathcal{L} + y_H\mathcal{R} + y_H \;=\; 0. \end{aligned} \tag{33}$$

The curve in the xy-plane described by this equation has a number of interesting properties. Besides passing through the orthogonal projection onto the xy-plane of the P3P solution points (including the camera's optical center), it also passes through the control points and the orthocenter of their triangle. It also has asymptotes that are perpendicular to the sides of this triangle. These claims can be checked directly, and are left as exercises for the reader. (The homogeneous version of Eq. (33) is useful here.) This curve will here be referred to as the *circumcubic*. The left side of the equation Eq. (33) will be denoted Γ.

The remainder of this section discusses other polynomials and curves concerning the P3P Problem and the setup used in this paper. Most of these are long and complicated, and seem to require algebraic manipulation software to handle. Quartic polynomials whose roots are the x and y coordinates of the P3P solution points are obtained in this manner. No proofs are offered here, but computer programs for all of this are available from the author upon request. Most of the work was accomplished using the *Mathematica* system[2], but certain long polynomial factorizations were achieved using the *Singular* system[3].

By studying the quantities S_j/η^2 $(j = 1,2,3)$, other curves in the xy-plane emerge. These also pass through the projections of the P3P solution points onto the xy-plane, and through the control points and their orthocenter. One initially obtains three seventh degree polynomials in x and y, whose coefficients can

[2] Wolfram Research, Inc.

[3] Department of Mathematics, University of Kaiserslautern

be expressed as polynomials in x_1, x_2, x_3, y_1, y_2, y_3, S_0, S_1, S_2 and S_3. Upon rescaling by η^2, the same may be said of the coefficients of the cubic polynomial Γ. By adding linear combinations of $x^4\Gamma$, $x^3y\Gamma$, $x^2y^2\Gamma$, $xy^3\Gamma$ and $y^4\Gamma$, to the seventh degree polynomials, three sixth degree polynomials can be obtained.

At this stage, it is helpful to introduce a rectangular hyperbolic curve, one that passes through the control points and their orthocenter. Let its equation be $Q = 0$. The three degree-6 polynomials, together with Γ times monomials of degree 3 or less, and Q times monomials of degree 4 or less, can be shown to be linearly dependent. The coefficient of Q in this linearly dependency is a degree-4 polynomial whose curve necessarily passes through the projections of the P3P solution points onto the xy-plane. In fact, rather surprisingly, the curve also passes through the control points and their orthocenter. It too has coefficients that can be expressed as polynomials in x_1, x_2, x_3, y_1, y_2, y_3, S_0, S_1, S_2 and S_3.

However, it is not unique, since the linear dependence can be accomplished in various ways. There is in fact a family of such quartic polynomials, resulting in a family of *circumquartic* curves, though any two of these polynomials differ by a linear combination of Γ, $x\Gamma$ and $y\Gamma$. One of these quartic polynomials is actually quadratic when regarded as a polynomial in y alone, and its curve has two vertical asymptotes.

We will focus attention on this particular circumquartic curve, and write its equation as $\Phi = 0$. An example can be seen in Figure 3, where the circumquartic is the lighter (solid) curve, consisting of four sections. The darker curve (blue, three sections) is the circumcubic. They intersect at various points, including the projections of the P3P solution points onto the xy-plane (dark, square dots), the control points (light dots) and their orthocenter (light dot).

The Weierstrass substitutions, $x_j = (1-t_j^2)/(1+t_j^2), y_j = 2t_j/(1+t_j^2)$ $(j = 1,2,3)$, are quite helpful here. Because of the particular coordinate system used in this paper, $t_1+t_2+t_3 = t_1t_2t_3$, and so, the coefficients of the polynomials in x and y here can be expressed as rational functions of t_2, t_3, S_0, S_1, S_2 and S_3. By rescaling the polynomials in x and y, these become polynomials whose coefficients are polynomials in t_2, t_3, S_0, S_1, S_2 and S_3. It is sometimes helpful, particularly when factoring polynomials and when computing a resultant of two polynomials in x and y, to replace the S_j with their expressions in terms of c_1, c_2 and c_3. In this way, the coefficients of polynomials in x and y are uniquely expressed as polynomials in t_2, t_3, c_1, c_2 and c_3.

It turns out that after making these adjustments, the resultant of the adjusted Γ and Φ that eliminates y, factors so that one of the factors is a quartic polynomial in x whose real roots are the x-coordinates of the P3P solution points. Likewise, a quartic polynomial in y whose real roots are the y-coordinates of the P3P solution points can be obtained. In Figure 3, the vertical (horizontal) dashed lines are simply the plotting of this quartic polynomial in x (in y). They both intersect the circumcubic and the circumquartic curves at the projections of the P3P control points.

The quartic polynomials in x (in y) is referred to as `xRes` (`yRes`) in the code that appears in the appendix. Admittedly, the formulas for these are quite complicated, but they have been thoroughly tested. It might be possible to obtain more pleasant expressions for these polynomials. This is suggested by some evidence, as follows. When the polynomials are rescaled by multiplying by $(1+t_2^2)^8(1+t_3^2)^8/(1-t_2t_3)^{16}$, the coefficients take on a certain symmetry, namely, they become invariant under permutations of the subscripts for the x_j, y_j, t_j, c_j and S_j (permuting these in the same way). Then, when the coefficients are expressed in terms of x_H, y_H, and the X_1, X_2, X_3, Y_1, Y_2, Y_3 and Δ that were introduced in Section 2 of (Rieck, 2018), rather than in terms of the t_j, many of the factors take on a simpler form. For instance, the factor `f123` at the end of the appendix in the present paper becomes the following:

$$\begin{aligned}&8503056\,\big[-1107-1188X_H+441X_H^2+464X_H^3\\&-113X_H^4-44X_H^5+11X_H^6+387Y_H^2+480X_HY_H^2\\&-50X_H^2Y_H^2-184X_H^3Y_H^2-25X_H^4Y_H^2-49Y_H^4\\&-76X_HY_H^4-35X_H^2Y_H^4+Y_H^6\big]\ /\\&\big[(3+X_2+X_H)^6\,(3+X_3+X_H)^6)\big].\end{aligned}$$

Some of the thus-adjusted factors have geometric significance, and some appear in (Rieck, 2018).

9 Conclusion

Three methods for solving the P3P Problem, developed by Finsterwalder, Grafarend-Lohse-Schaffrim, and Persson-Nordberg, all begin by deriving a cubic polynomial. When this is done in a general, more systematic way, a family of homogeneous cubic polynomials results. One of these has a special, well-studied inhomogeneous form Eq. (23), with rather simple coefficients, though two of these are complex numbers. Its occurrence in the context of studying the P3P Problem is new.

This polynomial's discriminant Eq. (6) vanishes if and only if a certain quantity (ζ_L) is on the standard deltoid curve, helping to explain the occurrence of deltoids, in a couple ways, in recent analyses of the P3P Problem. (See (Rieck, 2015), (Rieck, 2018), (Rieck-Wang, 2021), (Wang et al., 2022).) While the "deltoid phenomenon" in P3P is now better understood, via the new cubic polynomial, the existence of such a simple and useful polynomial is still surprising.

One of the coefficients of the new cubic polynomial (ζ_L again) was introduced in an earlier work (Rieck-Wang, 2021), where it was shown to have another useful form. This was restated as Theorem 3 in the present article. This article also further developed the "limit case" of the P3P Problem, by providing a far better geometric understanding of its solution points in space, in terms of triangles and deltoids.

Curves that pass through the orthogonal projections of the P3P solution points onto the xy-plane have also been obtained. These produce a quartic polynomial for the x coordinates of the P3P solution points. Similarly for their y coordinates.

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Appendix

A quartic polynomial (`xres`) whose real roots are the x-coordinates of the P3P solution points, and a quartic polynomial (`yres`) whose real roots are the y-coordinates of the P3P solution points, are detailed below.

```
xres = xco0 + xco1 * x + xco2 * x^2 + xco3 *
↪  x^3 + xco4 * x^4
```

```
yres = yco0 + yco1 * y + yco2 * y^2 + yco3 *
↪  y^3 + yco4 * y^4
```

where

```
xco0 = 8*f15^2*f16^2*f5^2*f53*f7^2*f8^2*S0 -
↪  4*f15^2*f16^2*f5^2*f53*f7^2*f8^2*S1 +
↪  4*f101*f15^2*f16^2*f7*f8*S0*S1 -
↪  f15^2*f16^2*f77*f78*S1^2 -
↪  4*f15^2*f16^2*f5^2*f53*f7^2*f8^2*S2 -
↪  4*f15^2*f5^2*f7*f8^3*f97*S0*S2 -
↪  2*f118*f15^2*f8^2*S1*S2 -
↪  16*f15^6*f2*f4*f7*f8^3*f9*S0*S1*S2 +
↪  4*f15^5*f77*f8^2*f9*S1^2*S2 +
↪  f15^2*f38*f42*f5^2*f8^4*S2^2 + 4*f15^5*f42*f8^4*S1*S2^2
↪  - 4*f15^2*f16^2*f5^2*f53*f7^2*f8^2*S3 -
↪  4*f16^2*f5^2*f7^3*f8*f98*S0*S3 -
↪  2*f119*f16^2*f7^2*S1*S3 -
↪  16*f1*f16^6*f3*f7^3*f8*f9*S0*S1*S3 +
↪  4*f16^5*f7^2*f78*f9*S1^2*S3 +
↪  2*f115*f5^2*f7^2*f8^2*S2*S3 +
↪  16*f13*f14*f5^6*f7^3*f8^3*f9*S0*S2*S3 +
↪  4*f122*f7^2*f8^2*S1*S2*S3 -
↪  16*f15^3*f16^3*f7^2*f8^2*f9^2*S1^2*S2*S3 -
↪  4*f38*f5^5*f7^2*f8^4*f9*S2^2*S3 -
↪  16*f15^3*f5^3*f7^2*f8^4*f9*S1*S2^2*S3 +
↪  f16^2*f37*f41*f5^2*f7^4*S3^2 + 4*f16^5*f41*f7^4*S1*S3^2
↪  + 4*f37*f5^5*f7^4*f8^2*f9*S2*S3^2 +
↪  16*f16^3*f5^3*f7^4*f8^2*f9*S1*S2*S3^2

xco1 = -64*f15^2*f16^2*f5^2*f7^3*f8^3*f9*S0 +
↪  32*f15^2*f16^2*f5^2*f7^3*f8^3*f9*S1 +
↪  16*f15^2*f16^2*f7^2*f76*f8^2*f9*S0*S1 -
↪  8*f10*f15^2*f16^2*f19*f6^2*f7^2*f8^2*f9*S1^2 +
↪  32*f15^2*f16^2*f5^2*f7^3*f8^3*f9*S2 +
↪  16*f15^2*f5^2*f7^2*f79*f8^3*S0*S2 -
↪  8*f106*f15^2*f7^2*f8^2*S1*S2 -
↪  8*f15^4*f7*f8^3*f86*S0*S1*S2 +
↪  4*f15^4*f7*f8^2*f95*S1^2*S2 -
↪  8*f12*f15^2*f28*f5^2*f7^2*f8^4*t2^2*S2^2 +
↪  4*f15^4*f58*f7*f8^4*S1*S2^2 +
↪  32*f15^2*f16^2*f5^2*f7^3*f8^3*f9*S3 +
↪  16*f16^2*f5^2*f7^3*f8^2*f80*S0*S3 -
↪  8*f107*f16^2*f7^2*f8^2*S1*S3 -
↪  8*f16^4*f7^3*f8*f85*S0*S1*S3 +
↪  4*f16^4*f7^2*f8*f96*S1^2*S3 -
↪  8*f112*f5^2*f7^2*f8^2*S2*S3 +
↪  8*f5^4*f63*f7^3*f8^3*f9*S0*S2*S3 +
↪  4*f123*f7^2*f8^2*S1*S2*S3 +
↪  32*f15^2*f16^2*f5^2*f7^3*f8^3*f9*S0*S1*S2*S3 -
↪  16*f15^2*f16^2*f48*f7^2*f8^2*f9*S1^2*S2*S3 -
↪  4*f5^4*f7^2*f71*f8^4*f9*S2^2*S3 -
↪  16*f15^2*f34*f5^2*f7^2*f8^4*f9*S1*S2^2*S3 -
↪  8*f11*f16^2*f26*f5^2*f7^4*f8^2*t3^2*S3^2 +
↪  4*f16^4*f59*f7^4*f8*S1*S3^2 -
↪  4*f5^4*f7^4*f70*f8^2*f9*S2*S3^2 -
↪  16*f16^2*f33*f5^2*f7^4*f8^2*f9*S1*S2*S3^2

xco2 = -16*f15^2*f16^2*f27*f5^2*f7^3*f8^3*S0 +
↪  8*f15^2*f16^2*f27*f5^2*f7^3*f8^3*S1 +
↪  8*f15^2*f16^2*f7^2*f8^2*f88*S0*S1 -
↪  2*f15^2*f16^2*f7^2*f8^2*f89*S1^2 +
↪  8*f15^2*f16^2*f27*f5^2*f7^3*f8^3*S2 +
↪  8*f15^2*f5^2*f7^2*f8^3*f93*S0*S2 -
↪  4*f110*f15^2*f7^2*f8^2*S1*S2 -
↪  16*f15^4*f47*f7^2*f8^3*S0*S1*S2 +
↪  4*f15^4*f62*f7^2*f8^3*S1^2*S2 -
↪  2*f15^2*f5^2*f68*f7^2*f8^4*S2^2 +
↪  4*f15^4*f46*f7^2*f8^4*S1*S2^2 +
↪  8*f15^2*f16^2*f27*f5^2*f7^3*f8^3*S3 +
↪  8*f16^2*f5^2*f7^3*f8^2*f94*S0*S3 -
↪  4*f111*f16^2*f7^2*f8^2*S1*S3 -
↪  16*f16^4*f47*f7^3*f8^2*S0*S1*S3 +
↪  4*f16^4*f61*f7^3*f8^2*S1^2*S3 -
↪  4*f116*f5^2*f7^2*f8^2*S2*S3 -
↪  16*f47*f5^4*f7^3*f8^3*f9^2*S0*S2*S3 +
↪  8*f121*f7^2*f8^2*S1*S2*S3 -
↪  16*f15^3*f16^3*f7^3*f8^3*S1^2*S2*S3 +
↪  4*f5^4*f54*f7^3*f8^4*f9*S2^2*S3 -
↪  16*f15^3*f5^3*f7^3*f8^4*f9*S1*S2^2*S3 -
↪  2*f16^2*f5^2*f69*f7^4*f8^2*S3^2 +
↪  4*f16^4*f45*f7^4*f8^2*S1*S3^2 +
↪  4*f5^4*f55*f7^4*f8^3*f9*S2*S3^2 +
↪  16*f16^3*f5^3*f7^4*f8^3*f9*S1*S2*S3^2

xco3 = -8*f15^4*f16*f5*f7^3*f8^4*S0*S1*S2 +
↪  4*f15^4*f16*f5*f7^3*f8^4*S1^2*S2 +
↪  4*f15^4*f16*f5*f7^3*f8^4*S1*S2^2 +
↪  8*f15*f16^4*f5*f7^4*f8^3*S0*S1*S3 -
↪  4*f15*f16^4*f5*f7^4*f8^3*S1^2*S3 +
↪  8*f15*f16*f5^4*f7^4*f8^4*f9*S0*S2*S3 +
↪  4*f15^2*f16^2*f29*f5^2*f7^3*f8^3*S1*S2*S3 -
↪  4*f15*f16*f5^4*f7^4*f8^4*f9*S2^2*S3 -
↪  4*f15*f16^4*f5*f7^4*f8^3*S1*S3^2 -
↪  4*f15*f16*f5^4*f7^4*f8^4*f9*S2*S3^2

yco0 = -32*f15^2*f16^2*f5^2*f6^2*f7^2*f8^2*t2^2*t3^2*S0 +
↪  16*f15^2*f16^2*f5^2*f6^2*f7^2*f8^2*t2^2*t3^2*S1 +
↪  32*f15^2*f16^2*f6^2*f60*f7*f8*t2*t3*S0*S1 -
↪  16*f15^2*f16^2*f35*f36*f6^2*S1^2 +
↪  16*f15^2*f16^2*f5^2*f6^2*f7^2*f8^2*t2^2*t3^2*S2 +
↪  32*f15^2*f5^2*f52*f6*f7*f8^3*t2^2*t3*S0*S2 -
↪  16*f109*f15^2*f6*f8^2*t2*S1*S2 -
↪  32*f15^6*f6*f7*f8^3*t2*t3^2*S0*S1*S2 +
↪  16*f15^5*f36*f6^2*f8^2*t3*S1^2*S2 -
↪  16*f15^2*f17*f31*f5^2*f8^4*t2^2*S2^2 +
↪  16*f15^5*f17*f8^4*t2^2*t3*S1*S2^2 +
↪  16*f15^2*f16^2*f5^2*f6^2*f7^2*f8^2*t2^2*t3^2*S3 +
↪  32*f16^2*f5^2*f51*f6*f7^3*f8*t2*t3^2*S0*S3 -
↪  16*f108*f16^2*f6*f7^2*t3*S1*S3 -
↪  32*f16^6*f6*f7^3*f8*t2^2*t3*S0*S1*S3 +
↪  16*f16^5*f35*f6^2*f7^2*t2*S1^2*S3 -
↪  16*f103*f5^2*f7^2*f8^2*t2*t3*S2*S3 -
↪  32*f5^6*f6^2*f7^3*f8^3*f9*t2*t3*S0*S2*S3 +
↪  16*f117*f7^2*f8^2*S1*S2*S3 -
↪  16*f15^3*f16^3*f6^2*f7^2*f8^2*t2*t3*S1^2*S2*S3 +
↪  16*f31*f5^5*f6*f7^2*f8^4*t2^2*S2^2*S3 -
↪  16*f15^3*f5^3*f6*f7^2*f8^4*t2^2*t3*S1*S2^2*S3 -
↪  16*f16^2*f18*f32*f5^2*f7^4*t3^2*S3^2 +
↪  16*f16^5*f18*f7^4*t2*t3^2*S1*S3^2 -
↪  16*f32*f5^5*f6*f7^4*f8^2*t3^2*S2*S3^2 +
↪  16*f16^3*f5^3*f6*f7^4*f8^2*t2*t3^2*S1*S2*S3^2
```

```
yco1 = 64*f15^2*f16^2*f5^2*f6*f7^3*f8^3*t2*t3*S0 -
↪  32*f15^2*f16^2*f5^2*f6*f7^3*f8^3*t2*t3*S1 -
↪  16*f15^2*f16^2*f6*f7^2*f73*f8^2*S0*S1 +
↪  16*f15^2*f16^2*f25*f30*f6*f7^2*f8^2*S1^2 -
↪  32*f15^2*f16^2*f5^2*f6*f7^3*f8^3*t2*t3*S2 -
↪  16*f15^2*f5^2*f7^2*f8^3*f82*t2*S0*S2 +
↪  16*f15^2*f7^2*f8^2*f99*t3*S1*S2 +
↪  16*f15^4*f7*f75*f8^3*t3*S0*S1*S2 -
↪  16*f15^4*f6*f7*f8^2*f83*t3*S1^2*S2 +
↪  16*f15^2*f21*f23*f5^2*f7^2*f8^4*t2*S2^2 -
↪  16*f15^4*f44*f7*f8^4*t2*t3*S1*S2^2 -
↪  32*f15^2*f16^2*f5^2*f6*f7^3*f8^3*t2*t3*S3 -
↪  16*f16^2*f5^2*f7^3*f8^2*f81*t3*S0*S3 +
↪  16*f100*f16^2*f7^2*f8^2*t2*S1*S3 +
↪  16*f16^4*f7^3*f74*f8*t2*S0*S1*S3 -
↪  16*f16^4*f6*f7^2*f8*f84*t2*S1^2*S3 +
↪  16*f102*f5^2*f6*f7^2*f8^2*S2*S3 +
↪  16*f5^4*f6*f7^3*f72*f8^3*S0*S2*S3 -
↪  16*f114*f6*f7^2*f8^2*t2*t3*S1*S2*S3 -
↪  32*f15^2*f16^2*f5^2*f6*f7^3*f8^3*t2*t3*S0*S1*S2*S3 +
↪  16*f15^2*f16^2*f48*f6*f7^2*f8^2*t2*t3*S1^2*S2*S3 -
↪  16*f5^4*f57*f6*f7^2*f8^4*t2*S2^2*S3 +
↪  16*f15^2*f34*f5^2*f6*f7^2*f8^4*t2*t3*S1*S2^2*S3 +
↪  16*f16^2*f20*f24*f5^2*f7^4*f8^2*t3*S3^2 -
↪  16*f16^4*f43*f7^4*f8*t2*t3*S1*S3^2 -
↪  16*f5^4*f56*f6*f7^4*f8^2*t3*S2*S3^2 +
↪  16*f16^2*f33*f5^2*f6*f7^4*f8^2*t2*t3*S1*S2*S3^2
```

```
yco2 = -32*f15^2*f16^2*f22*f5^2*f7^3*f8^3*S0 +
↪  16*f15^2*f16^2*f22*f5^2*f7^3*f8^3*S1 +
↪  8*f15^2*f16^2*f7^2*f8^2*f90*S0*S1 -
↪  4*f15^2*f16^2*f7^2*f8^2*f87*S1^2 +
↪  16*f15^2*f16^2*f22*f5^2*f7^3*f8^3*S2 +
↪  8*f15^2*f5^2*f7^2*f8^3*f91*S0*S2 -
↪  4*f104*f15^2*f7^2*f8^2*S1*S2 -
↪  16*f15^4*f47*f7^2*f8^3*t3^2*S0*S1*S2 +
↪  4*f15^4*f66*f7^2*f8^3*t3*S1^2*S2 -
↪  4*f15^2*f5^2*f64*f7^2*f8^4*S2^2 +
↪  4*f15^4*f40*f7^2*f8^4*t3*S1*S2^2 +
↪  16*f15^2*f16^2*f22*f5^2*f7^3*f8^3*S3 +
↪  8*f16^2*f5^2*f7^3*f8^2*f92*S0*S3 -
↪  4*f105*f16^2*f7^2*f8^2*S1*S3 -
↪  16*f16^4*f47*f7^3*f8^2*t2^2*S0*S1*S3 +
↪  4*f16^4*f67*f7^3*f8^2*t2*S1^2*S3 -
↪  4*f113*f5^2*f7^2*f8^2*S2*S3 -
↪  16*f47*f5^4*f6^2*f7^3*f8^3*S0*S2*S3 +
↪  8*f120*f7^2*f8^2*S1*S2*S3 -
↪  16*f15^3*f16^3*f7^3*f8^3*t2*t3*S1^2*S2*S3 +
↪  4*f5^4*f50*f6*f7^3*f8^4*S2^2*S3 -
↪  16*f15^3*f5^3*f6*f7^3*f8^4*t3*S1*S2^2*S3 -
↪  4*f16^2*f5^2*f65*f7^4*f8^2*S3^2 +
↪  4*f16^4*f39*f7^4*f8^2*t2*S1*S3^2 +
↪  4*f49*f5^4*f6*f7^4*f8^3*S2*S3^2 +
↪  16*f16^3*f5^3*f6*f7^4*f8^3*t2*S1*S2*S3^2
```

```
yco3 = 8*f15^4*f16*f5*f7^3*f8^4*t3*S0*S1*S2 -
↪  4*f15^4*f16*f5*f7^3*f8^4*t3*S1^2*S2 -
↪  4*f15^4*f16*f5*f7^3*f8^4*t3*S1*S2^2 -
↪  8*f15*f16^4*f5*f7^4*f8^3*t2*S0*S1*S3 +
↪  4*f15*f16^4*f5*f7^4*f8^3*t2*S1^2*S3 -
↪  8*f15*f16*f5^4*f6*f7^4*f8^4*S0*S2*S3 -
↪  16*f15^2*f16^2*f5^2*f6*f7^3*f8^3*t2*t3*S1*S2*S3 +
↪  4*f15*f16*f5^4*f6*f7^4*f8^4*S2^2*S3 +
↪  4*f15*f16^4*f5*f7^4*f8^3*t2*S1*S3^2 +
↪  4*f15*f16*f5^4*f6*f7^4*f8^4*S2*S3^2
```

```
f1 = 1 + t2

f2 = 1 + t3

f3 = -1 + t2

f4 = -1 + t3

f5 = t2 - t3

f6 = t2 + t3

f7 = 1 + t2^2

f8 = 1 + t3^2

f9 = -1 + t2*t3

f10 = -3 + t2*t3

f11 = -3 - t2^2 + 2*t2*t3

f12 = -3 + 2*t2*t3 - t3^2

f13 = -1 - t2 - t3 + t2*t3

f14 = -1 + t2 + t3 + t2*t3

f15 = -2*t2 - t3 + t2^2*t3

f16 = -t2 - 2*t3 + t2*t3^2

f17 = t2^4 - 2*t2*t3 - t3^2

f18 = -t2^2 - 2*t2*t3 + t3^4

f19 = -3 - t2^2 - t3^2 + t2^2*t3^2

f20 = -t2^2 - t2*t3 - t3^2 + t2*t3^3

f21 = -t2^2 - t2*t3 + t2^3*t3 - t3^2

f22 = t2^2 + t2*t3 + t3^2 + t2^2*t3^2

f23 = t2^2 - 2*t2*t3 - 2*t3^2 + t2^2*t3^2

f24 = -2*t2^2 - 2*t2*t3 + t3^2 + t2^2*t3^2

f25 = -t2^2 - 4*t2*t3 - t3^2 + 2*t2^2*t3^2

f26 = 3 - t2^2 - 4*t2*t3 + t3^2 + t2^2*t3^2
```

```
f27 = 3 + t2^2 - 2*t2*t3 + t3^2 + t2^2*t3^2

f28 = 3 + t2^2 - 4*t2*t3 - t3^2 + t2^2*t3^2

f29 = -3 + t2^2 + 4*t2*t3 + t3^2 + t2^2*t3^2

f30 = -t2^2 - t2*t3 - t3^2 - 2*t2^2*t3^2 + t2^3*t3^3

f31 = t2^2 - 2*t2^3*t3 - t3^2 - t2^2*t3^2 + t2^4*t3^2

f32 = -t2^2 + t3^2 - t2^2*t3^2 - 2*t2*t3^3 + t2^2*t3^4

f33 = 2*t2^2 + 2*t2*t3 + 5*t3^2 - t2^2*t3^2 - 6*t2*t3^3 +
↪  t3^4 + t2^2*t3^4

f34 = 5*t2^2 + t2^4 + 2*t2*t3 - 6*t2^3*t3 + 2*t3^2 -
↪  t2^2*t3^2 + t2^4*t3^2

f35 = -t2^2 - 2*t2*t3 - 4*t2*t3^3 - 2*t2^3*t3^3 - t3^4 +
↪  t2^2*t3^4 + t2^4*t3^4

f36 = -t2^4 - 2*t2*t3 - 4*t2^3*t3 - t3^2 + t2^4*t3^2 -
↪  2*t2^3*t3^3 + t2^4*t3^4

f37 = 2*t2 + 7*t3 + t2^2*t3 + 4*t2*t3^2 + 10*t3^3 -
↪  2*t2^2*t3^3 - 6*t2*t3^4 - t3^5 + t2^2*t3^5

f38 = 7*t2 + 10*t2^3 - t2^5 + 2*t3 + 4*t2^2*t3 - 6*t2^4*t3
↪  + t2*t3^2 - 2*t2^3*t3^2 + t2^5*t3^2

f39 = 4*t2^3 + 2*t2^2*t3 + 5*t2*t3^2 - t2^3*t3^2 - 2*t3^3 -
↪  8*t2^2*t3^3 + 3*t2*t3^4 + t2^3*t3^4

f40 = -2*t2^3 + 5*t2^2*t3 + 3*t2^4*t3 + 2*t2*t3^2 -
↪  8*t2^3*t3^2 + 4*t3^3 - t2^2*t3^3 + t2^4*t3^3

f41 = 2*t2 - 5*t3 - 3*t2^2*t3 + 12*t2*t3^2 - 2*t3^3 -
↪  6*t2^2*t3^3 + 2*t2*t3^4 - t3^5 + t2^2*t3^5

f42 = -5*t2 - 2*t2^3 - t2^5 + 2*t3 + 12*t2^2*t3 + 2*t2^4*t3
↪  - 3*t2*t3^2 - 6*t2^3*t3^2 + t2^5*t3^2

f43 = 2*t2^3 + 4*t2^2*t3 + t2*t3^2 - t2^3*t3^2 + 2*t3^3 -
↪  2*t2^2*t3^3 - 4*t2*t3^4 + t3^5 + t2^2*t3^5

f44 = 2*t2^3 + t2^5 + t2^2*t3 - 4*t2^4*t3 + 4*t2*t3^2 -
↪  2*t2^3*t3^2 + t2^5*t3^2 + 2*t3^3 - t2^2*t3^3

f45 = 2*t2^2 - t2*t3 + t2^3*t3 + 8*t3^2 + 2*t2^2*t3^2 -
↪  11*t2*t3^3 - t2^3*t3^3 + 2*t3^4 + 2*t2^2*t3^4

f46 = 8*t2^2 + 2*t2^4 - t2*t3 - 11*t2^3*t3 + 2*t3^2 +
↪  2*t2^2*t3^2 + 2*t2^4*t3^2 + t2*t3^3 - t2^3*t3^3

f47 = 3*t2^2 + t2^4 + 3*t2*t3 - t2^3*t3 + 3*t3^2 +
↪  3*t2^2*t3^2 - t2*t3^3 - 5*t2^3*t3^3 + t3^4 + t2^4*t3^4

f48 = 5*t2^2 + t2^4 + 8*t2*t3 + 5*t3^2 + 4*t2^2*t3^2 -
↪  t2^4*t3^2 - 8*t2^3*t3^3 + t3^4 - t2^2*t3^4 +
↪  2*t2^4*t3^4

f49 = 4*t2^3 + 10*t2^2*t3 + 13*t2*t3^2 - 3*t2^3*t3^2 +
↪  9*t3^3 - 11*t2^2*t3^3 - 11*t2*t3^4 + t2^3*t3^4 + t3^5 +
↪  3*t2^2*t3^5

f50 = 9*t2^3 + t2^5 + 13*t2^2*t3 - 11*t2^4*t3 + 10*t2*t3^2
↪  - 11*t2^3*t3^2 + 3*t2^5*t3^2 + 4*t3^3 - 3*t2^2*t3^3 +
↪  t2^4*t3^3

f51 = t2^2 + t2^4 + t2*t3 + t2^3*t3 - 2*t3^2 + 2*t2^2*t3^2
↪  + 6*t2*t3^3 - 2*t2^3*t3^3 - t3^4 - 5*t2^2*t3^4 +
↪  t2*t3^5 + t2^3*t3^5

f52 = -2*t2^2 - t2^4 + t2*t3 + 6*t2^3*t3 + t2^5*t3 + t3^2 +
↪  2*t2^2*t3^2 - 5*t2^4*t3^2 + t2*t3^3 - 2*t2^3*t3^3 +
↪  t2^5*t3^3 + t3^4

f53 = -3 - 2*t2^2 + t2^4 + 4*t2*t3 + 4*t2^3*t3 - 2*t3^2 -
↪  2*t2^4*t3^2 + 4*t2*t3^3 - 4*t2^3*t3^3 + t3^4 -
↪  2*t2^2*t3^4 + t2^4*t3^4

f54 = -11*t2^2 - 3*t2^4 - 5*t2*t3 + 21*t2^3*t3 + 2*t2^5*t3
↪  - 2*t3^2 + 9*t2^2*t3^2 - 13*t2^4*t3^2 + 3*t2*t3^3 -
↪  3*t2^3*t3^3 + 2*t2^5*t3^3

f55 = -2*t2^2 - 5*t2*t3 + 3*t2^3*t3 - 11*t3^2 + 9*t2^2*t3^2
↪  + 21*t2*t3^3 - 3*t2^3*t3^3 - 3*t3^4 - 13*t2^2*t3^4 +
↪  2*t2*t3^5 + 2*t2^3*t3^5

f56 = 2*t2^3 + 2*t2^2*t3 - t2*t3^2 + t2^3*t3^2 - 3*t3^3 +
↪  3*t2^2*t3^3 + 6*t2*t3^4 - 2*t2^3*t3^4 - t3^5 -
↪  5*t2^2*t3^5 + t2*t3^6 + t2^3*t3^6

f57 = -3*t2^3 - t2^5 - t2^2*t3 + 6*t2^4*t3 + t2^6*t3 +
↪  2*t2*t3^2 + 3*t2^3*t3^2 - 5*t2^5*t3^2 + 2*t3^3 +
↪  t2^2*t3^3 - 2*t2^4*t3^3 + t2^6*t3^3

f58 = -17*t2^2 - 6*t2^4 - t2^6 + t2*t3 + 34*t2^3*t3 +
↪  5*t2^5*t3 - 2*t3^2 - 3*t2^2*t3^2 - 20*t2^4*t3^2 +
↪  t2^6*t3^2 + 5*t2*t3^3 + 2*t2^3*t3^3 + t2^5*t3^3

f59 = -2*t2^2 + t2*t3 + 5*t2^3*t3 - 17*t3^2 - 3*t2^2*t3^2 +
↪  34*t2*t3^3 + 2*t2^3*t3^3 - 6*t3^4 - 20*t2^2*t3^4 +
↪  5*t2*t3^5 + t2^3*t3^5 - t3^6 + t2^2*t3^6

f60 = 2*t2^2 + t2^4 + 5*t2*t3 + 4*t2^3*t3 + 2*t3^2 +
↪  2*t2^2*t3^2 - 2*t2^4*t3^2 + 4*t2*t3^3 + 2*t2^3*t3^3 +
↪  t3^4 - 2*t2^2*t3^4 - 4*t2^4*t3^4 + t2^5*t3^5

f61 = 11*t2^2 + 3*t2^4 + 17*t2*t3 - t2^5*t3 + 8*t3^2 +
↪  4*t2^2*t3^2 - 4*t2^4*t3^2 + 3*t2*t3^3 - 12*t2^3*t3^3 +
↪  t2^5*t3^3 + 2*t3^4 - 3*t2^2*t3^4 + 3*t2^4*t3^4

f62 = 8*t2^2 + 2*t2^4 + 17*t2*t3 + 3*t2^3*t3 + 11*t3^2 +
↪  4*t2^2*t3^2 - 3*t2^4*t3^2 - 12*t2^3*t3^3 + 3*t3^4 -
↪  4*t2^2*t3^4 + 3*t2^4*t3^4 - t2*t3^5 + t2^3*t3^5

f63 = -10*t2^2 - 2*t2^4 - 7*t2*t3 - t2^5*t3 - 10*t3^2 -
↪  32*t2^2*t3^2 + 2*t2^4*t3^2 + 40*t2^3*t3^3 - 2*t3^4 +
↪  2*t2^2*t3^4 - 12*t2^4*t3^4 - t2*t3^5 + t2^5*t3^5

f64 = 3*t2^4 + t2^6 + 4*t2^3*t3 - 4*t2^5*t3 + 8*t2^2*t3^2 -
↪  2*t2^4*t3^2 + 2*t2^6*t3^2 + 8*t2*t3^3 - 4*t2^3*t3^3 -
↪  4*t2^5*t3^3 + 4*t3^4 - t2^4*t3^4 + t2^6*t3^4
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f65 = 4*t2^4 + 8*t2^3*t3 + 8*t2^2*t3^2 + 4*t2*t3^3 -
↪  4*t2^3*t3^3 + 3*t3^4 - 2*t2^2*t3^4 - t2^4*t3^4 -
↪  4*t2*t3^5 - 4*t2^3*t3^5 + t3^6 + 2*t2^2*t3^6 +
↪  t2^4*t3^6

f66 = 2*t2^3 + 11*t2^2*t3 + t2^4*t3 + 14*t2*t3^2 +
↪  2*t2^3*t3^2 + 9*t3^3 + 10*t2^2*t3^3 - 3*t2^4*t3^3 -
↪  2*t2*t3^4 - 16*t2^3*t3^4 + t3^5 - t2^2*t3^5 +
↪  4*t2^4*t3^5

f67 = 9*t2^3 + t2^5 + 14*t2^2*t3 - 2*t2^4*t3 + 11*t2*t3^2 +
↪  10*t2^3*t3^2 - t2^5*t3^2 + 2*t3^3 + 2*t2^2*t3^3 -
↪  16*t2^4*t3^3 + t2*t3^4 - 3*t2^3*t3^4 + 4*t2^5*t3^4

f68 = 19*t2^2 + 16*t2^4 + t2^6 + 4*t2*t3 - 32*t2^3*t3 -
↪  20*t2^5*t3 + 4*t3^2 + 2*t2^2*t3^2 + 28*t2^4*t3^2 +
↪  6*t2^6*t3^2 - 4*t2*t3^3 - 12*t2^5*t3^3 + 3*t2^2*t3^4 +
↪  t2^6*t3^4

f69 = 4*t2^2 + 4*t2*t3 - 4*t2^3*t3 + 19*t3^2 + 2*t2^2*t3^2
↪  + 3*t2^4*t3^2 - 32*t2*t3^3 + 16*t3^4 + 28*t2^2*t3^4 -
↪  20*t2*t3^5 - 12*t2^3*t3^5 + t3^6 + 6*t2^2*t3^6 +
↪  t2^4*t3^6

f70 = -2*t2^2 - 5*t2*t3 - 3*t2^3*t3 - 20*t3^2 -
↪  10*t2^2*t3^2 + 5*t2*t3^3 + 3*t2^3*t3^3 - 28*t3^4 +
↪  6*t2^2*t3^4 + 33*t2*t3^5 - t2^3*t3^5 - 10*t2^2*t3^6 -
↪  t2*t3^7 + t2^3*t3^7

f71 = -20*t2^2 - 28*t2^4 - 5*t2*t3 + 5*t2^3*t3 + 33*t2^5*t3
↪  - t2^7*t3 - 2*t3^2 - 10*t2^2*t3^2 + 6*t2^4*t3^2 -
↪  10*t2^6*t3^2 - 3*t2*t3^3 + 3*t2^3*t3^3 - t2^5*t3^3 +
↪  t2^7*t3^3

f72 = -2*t2^4 + t2^3*t3 + 3*t2^5*t3 + 2*t2^2*t3^2 +
↪  2*t2^4*t3^2 + t2*t3^3 + 8*t2^3*t3^3 - t2^5*t3^3 -
↪  2*t3^4 + 2*t2^2*t3^4 - 10*t2^4*t3^4 + 3*t2*t3^5 -
↪  t2^3*t3^5 + 2*t2^5*t3^5

f73 = 2*t2^4 + 11*t2^3*t3 + t2^5*t3 + 10*t2^2*t3^2 -
↪  2*t2^4*t3^2 + 11*t2*t3^3 + 14*t2^3*t3^3 - t2^5*t3^3 +
↪  2*t3^4 - 2*t2^2*t3^4 - 18*t2^4*t3^4 + t2*t3^5 -
↪  t2^3*t3^5 + 4*t2^5*t3^5

f74 = 2*t2^4 + 9*t2^3*t3 + t2^5*t3 + 13*t2^2*t3^2 -
↪  t2^4*t3^2 + 8*t2*t3^3 + 4*t3^4 + 8*t2^2*t3^4 -
↪  6*t2^4*t3^4 + 2*t2*t3^5 - 13*t2^3*t3^5 + t2^5*t3^5 +
↪  2*t3^6 - t2^2*t3^6 + 3*t2^4*t3^6

f75 = 4*t2^4 + 2*t2^6 + 8*t2^3*t3 + 2*t2^5*t3 +
↪  13*t2^2*t3^2 + 8*t2^4*t3^2 - t2^6*t3^2 + 9*t2*t3^3 -
↪  13*t2^5*t3^3 + 2*t3^4 - t2^2*t3^4 - 6*t2^4*t3^4 +
↪  3*t2^6*t3^4 + t2*t3^5 + t2^5*t3^5

f76 = 10*t2^2 + 4*t2^4 + 16*t2*t3 + t2^3*t3 - t2^5*t3 +
↪  10*t3^2 + 2*t2^2*t3^2 - 4*t2^4*t3^2 + t2*t3^3 -
↪  10*t2^3*t3^3 + t2^5*t3^3 + 4*t3^4 - 4*t2^2*t3^4 +
↪  2*t2^4*t3^4 - t2*t3^5 + t2^3*t3^5

f77 = 5*t2 + 2*t2^3 + t2^5 + 7*t3 - 2*t2^2*t3 + 3*t2^4*t3 -
↪  2*t2*t3^2 + 4*t2^3*t3^2 - 2*t2^5*t3^2 + 10*t3^3 -
↪  4*t2^2*t3^3 - 6*t2^4*t3^3 - 19*t2*t3^4 - 6*t2^3*t3^4 +
↪  t2^5*t3^4 - t3^5 + 6*t2^2*t3^5 + 3*t2^4*t3^5

f78 = 7*t2 + 10*t2^3 - t2^5 + 5*t3 - 2*t2^2*t3 - 19*t2^4*t3
↪  - 2*t2*t3^2 - 4*t2^3*t3^2 + 6*t2^5*t3^2 + 2*t3^3 +
↪  4*t2^2*t3^3 - 6*t2^4*t3^3 + 3*t2*t3^4 - 6*t2^3*t3^4 +
↪  3*t2^5*t3^4 + t3^5 - 2*t2^2*t3^5 + t2^4*t3^5

f79 = -10*t2^2 - 4*t2^4 - 4*t2*t3 + 15*t2^3*t3 + 3*t2^5*t3
↪  - 4*t3^2 - 7*t2^2*t3^2 - 9*t2^4*t3^2 + 8*t2*t3^3 +
↪  26*t2^3*t3^3 + 2*t2^5*t3^3 - 4*t2^2*t3^4 - 18*t2^4*t3^4
↪  + 3*t2^3*t3^5 + 3*t2^5*t3^5 + t2^2*t3^6 - t2^4*t3^6

f80 = -4*t2^2 - 4*t2*t3 + 8*t2^3*t3 - 10*t3^2 - 7*t2^2*t3^2
↪  - 4*t2^4*t3^2 + t2^6*t3^2 + 15*t2*t3^3 + 26*t2^3*t3^3 +
↪  3*t2^5*t3^3 - 4*t3^4 - 9*t2^2*t3^4 - 18*t2^4*t3^4 -
↪  t2^6*t3^4 + 3*t2*t3^5 + 2*t2^3*t3^5 + 3*t2^5*t3^5

f81 = 8*t2^4 + 4*t2^6 + 16*t2^3*t3 + 4*t2^5*t3 +
↪  11*t2^2*t3^2 + 2*t2^4*t3^2 - t2^6*t3^2 + 3*t2*t3^3 +
↪  2*t2^3*t3^3 - 9*t2^5*t3^3 - 2*t3^4 - 3*t2^2*t3^4 -
↪  12*t2^4*t3^4 + t2^6*t3^4 + 3*t2*t3^5 + 2*t2^3*t3^5 +
↪  3*t2^5*t3^5

f82 = -2*t2^4 + 3*t2^3*t3 + 3*t2^5*t3 + 11*t2^2*t3^2 -
↪  3*t2^4*t3^2 + 16*t2*t3^3 + 2*t2^3*t3^3 + 2*t2^5*t3^3 +
↪  8*t3^4 + 2*t2^2*t3^4 - 12*t2^4*t3^4 + 4*t2*t3^5 -
↪  9*t2^3*t3^5 + 3*t2^5*t3^5 + 4*t3^6 - t2^2*t3^6 +
↪  t2^4*t3^6

f83 = 2*t2^3 + t2^5 + 5*t2^2*t3 + 5*t2^4*t3 + 8*t2*t3^2 +
↪  13*t2^3*t3^2 - 2*t2^5*t3^2 + 3*t3^3 + 2*t2^2*t3^3 -
↪  9*t2^4*t3^3 + 2*t2*t3^4 + 6*t2^3*t3^4 + t2^5*t3^4 +
↪  t3^5 + t2^2*t3^5 - 8*t2^4*t3^5 - t2^3*t3^6 +
↪  2*t2^5*t3^6

f84 = 3*t2^3 + t2^5 + 8*t2^2*t3 + 2*t2^4*t3 + 5*t2*t3^2 +
↪  2*t2^3*t3^2 + t2^5*t3^2 + 2*t3^3 + 13*t2^2*t3^3 +
↪  6*t2^4*t3^3 - t2^6*t3^3 + 5*t2*t3^4 - 9*t2^3*t3^4 -
↪  8*t2^5*t3^4 + t3^5 - 2*t2^2*t3^5 + t2^4*t3^5 +
↪  2*t2^6*t3^5

f85 = -10*t2^2 - 2*t2^4 - 13*t2*t3 + 22*t2^3*t3 + 3*t2^5*t3
↪  - 13*t3^2 + 8*t2^2*t3^2 - 11*t2^4*t3^2 + 6*t2*t3^3 +
↪  24*t2^3*t3^3 + 2*t2^5*t3^3 - 10*t3^4 + 10*t2^2*t3^4 -
↪  20*t2^4*t3^4 + 7*t2*t3^5 - 6*t2^3*t3^5 + 3*t2^5*t3^5 -
↪  t3^6 + t2^4*t3^6

f86 = -13*t2^2 - 10*t2^4 - t2^6 - 13*t2*t3 + 6*t2^3*t3 +
↪  7*t2^5*t3 - 10*t3^2 + 8*t2^2*t3^2 + 10*t2^4*t3^2 +
↪  22*t2*t3^3 + 24*t2^3*t3^3 - 6*t2^5*t3^3 - 2*t3^4 -
↪  11*t2^2*t3^4 - 20*t2^4*t3^4 + t2^6*t3^4 + 3*t2*t3^5 +
↪  2*t2^3*t3^5 + 3*t2^5*t3^5

f87 = 3*t2^4 + t2^6 + 8*t2^3*t3 + 4*t2^5*t3 + 14*t2^2*t3^2
↪  + 15*t2^4*t3^2 - t2^6*t3^2 + 8*t2*t3^3 + 8*t2^3*t3^3 -
↪  8*t2^5*t3^3 + 3*t3^4 + 15*t2^2*t3^4 + 10*t2^4*t3^4 +
↪  4*t2*t3^5 - 8*t2^3*t3^5 - 16*t2^5*t3^5 + t3^6 -
↪  t2^2*t3^6 + 4*t2^6*t3^6

f88 = 11*t2^2 + 10*t2^4 + t2^6 + 14*t2*t3 + 5*t2^3*t3 -
↪  5*t2^5*t3 + 11*t3^2 + 18*t2^2*t3^2 + 6*t2^4*t3^2 +
↪  t2^6*t3^2 + 5*t2*t3^3 - 12*t2^3*t3^3 - 9*t2^5*t3^3 +
↪  10*t3^4 + 6*t2^2*t3^4 + 2*t2^6*t3^4 - 5*t2*t3^5 -
↪  9*t2^3*t3^5 + t3^6 + t2^2*t3^6 + 2*t2^4*t3^6
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f89 = 19*t2^2 + 16*t2^4 + t2^6 + 34*t2*t3 + 20*t2^3*t3 -
↪  6*t2^5*t3 + 19*t3^2 + 24*t2^2*t3^2 + t2^4*t3^2 +
↪  20*t2*t3^3 - 8*t2^3*t3^3 - 12*t2^5*t3^3 + 16*t3^4 +
↪  t2^2*t3^4 - 8*t2^4*t3^4 + 3*t2^6*t3^4 - 6*t2*t3^5 -
↪  12*t2^3*t3^5 + 2*t2^5*t3^5 + t3^6 + 3*t2^4*t3^6

f90 = 4*t2^4 + 2*t2^6 + 7*t2^3*t3 + 3*t2^5*t3 +
↪  14*t2^2*t3^2 + 17*t2^4*t3^2 + t2^6*t3^2 + 7*t2*t3^3 +
↪  4*t2^3*t3^3 - 11*t2^5*t3^3 + 4*t3^4 + 17*t2^2*t3^4 +
↪  12*t2^4*t3^4 + t2^6*t3^4 + 3*t2*t3^5 - 11*t2^3*t3^5 -
↪  18*t2^5*t3^5 + 2*t3^6 + t2^2*t3^6 + t2^4*t3^6 +
↪  4*t2^6*t3^6

f91 = 4*t2^4 + 2*t2^6 + 9*t2^3*t3 + t2^5*t3 + 17*t2^2*t3^2
↪  + 9*t2^4*t3^2 + 2*t2^6*t3^2 + 16*t2*t3^3 + 6*t2^3*t3^3
↪  - 10*t2^5*t3^3 + 8*t3^4 + 10*t2^2*t3^4 - 6*t2^4*t3^4 +
↪  2*t2^6*t3^4 + 4*t2*t3^5 - 11*t2^3*t3^5 - 7*t2^5*t3^5 +
↪  4*t3^6 + t2^2*t3^6 + t2^4*t3^6 + 2*t2^6*t3^6

f92 = 8*t2^4 + 4*t2^6 + 16*t2^3*t3 + 4*t2^5*t3 +
↪  17*t2^2*t3^2 + 10*t2^4*t3^2 + t2^6*t3^2 + 9*t2*t3^3 +
↪  6*t2^3*t3^3 - 11*t2^5*t3^3 + 4*t3^4 + 9*t2^2*t3^4 -
↪  6*t2^4*t3^4 + t2^6*t3^4 + t2*t3^5 - 10*t2^3*t3^5 -
↪  7*t2^5*t3^5 + 2*t3^6 + 2*t2^2*t3^6 + 2*t2^4*t3^6 +
↪  2*t2^6*t3^6

f93 = 11*t2^2 + 10*t2^4 + t2^6 + 8*t2*t3 - 9*t2^3*t3 -
↪  9*t2^5*t3 + 8*t3^2 + 12*t2^2*t3^2 + 17*t2^4*t3^2 +
↪  3*t2^6*t3^2 - 8*t2*t3^3 - 22*t2^3*t3^3 - 14*t2^5*t3^3 +
↪  4*t3^4 + 11*t2^2*t3^4 + 20*t2^4*t3^4 + 3*t2^6*t3^4 -
↪  4*t2*t3^5 - 5*t2^3*t3^5 - 9*t2^5*t3^5 + 2*t2^2*t3^6 +
↪  t2^4*t3^6 + t2^6*t3^6

f94 = 8*t2^2 + 4*t2^4 + 8*t2*t3 - 8*t2^3*t3 - 4*t2^5*t3 +
↪  11*t3^2 + 12*t2^2*t3^2 + 11*t2^4*t3^2 + 2*t2^6*t3^2 -
↪  9*t2*t3^3 - 22*t2^3*t3^3 - 5*t2^5*t3^3 + 10*t3^4 +
↪  17*t2^2*t3^4 + 20*t2^4*t3^4 + t2^6*t3^4 - 9*t2*t3^5 -
↪  14*t2^3*t3^5 - 9*t2^5*t3^5 + t3^6 + 3*t2^2*t3^6 +
↪  3*t2^4*t3^6 + t2^6*t3^6

f95 = -17*t2^2 - 6*t2^4 - t2^6 - 35*t2*t3 + 10*t2^3*t3 +
↪  t2^5*t3 - 20*t3^2 + 17*t2^2*t3^2 - 6*t2^4*t3^2 +
↪  t2^6*t3^2 - 17*t2*t3^3 + 14*t2^3*t3^3 + 11*t2^5*t3^3 -
↪  28*t3^4 + 53*t2^2*t3^4 + 18*t2^4*t3^4 - 3*t2^6*t3^4 +
↪  51*t2*t3^5 + 6*t2^3*t3^5 - 17*t2^5*t3^5 - 21*t2^2*t3^6
↪  - 22*t2^4*t3^6 + 3*t2^6*t3^6 + t2*t3^7 + 2*t2^3*t3^7 +
↪  5*t2^5*t3^7

f96 = -20*t2^2 - 28*t2^4 - 35*t2*t3 - 17*t2^3*t3 +
↪  51*t2^5*t3 + t2^7*t3 - 17*t3^2 + 17*t2^2*t3^2 +
↪  53*t2^4*t3^2 - 21*t2^6*t3^2 + 10*t2*t3^3 + 14*t2^3*t3^3
↪  + 6*t2^5*t3^3 + 2*t2^7*t3^3 - 6*t3^4 - 6*t2^2*t3^4 +
↪  18*t2^4*t3^4 - 22*t2^6*t3^4 + t2*t3^5 + 11*t2^3*t3^5 -
↪  17*t2^5*t3^5 + 5*t2^7*t3^5 - t3^6 + t2^2*t3^6 -
↪  3*t2^4*t3^6 + 3*t2^6*t3^6

f97 = -19*t2^2 - 17*t2^4 + 3*t2^6 + t2^8 - 4*t2*t3 +
↪  30*t2^3*t3 + 40*t2^5*t3 - 2*t2^7*t3 - 4*t3^2 -
↪  15*t2^2*t3^2 - 7*t2^4*t3^2 - 37*t2^6*t3^2 - t2^8*t3^2 +
↪  16*t2*t3^3 + 60*t2^3*t3^3 - 24*t2^5*t3^3 + 12*t2^7*t3^3
↪  + 4*t3^4 - 5*t2^2*t3^4 - 59*t2^4*t3^4 + 29*t2^6*t3^4 -
↪  t2^8*t3^4 - 4*t2*t3^5 - 10*t2^3*t3^5 + 24*t2^5*t3^5 -
↪  10*t2^7*t3^5 - t2^2*t3^6 + 3*t2^4*t3^6 - 3*t2^6*t3^6 +
↪  t2^8*t3^6

f98 = -4*t2^2 + 4*t2^4 - 4*t2*t3 + 16*t2^3*t3 - 4*t2^5*t3 -
↪  19*t3^2 - 15*t2^2*t3^2 - 5*t2^4*t3^2 - t2^6*t3^2 +
↪  30*t2*t3^3 + 60*t2^3*t3^3 - 10*t2^5*t3^3 - 17*t3^4 -
↪  7*t2^2*t3^4 - 59*t2^4*t3^4 + 3*t2^6*t3^4 + 40*t2*t3^5 -
↪  24*t2^3*t3^5 + 24*t2^5*t3^5 + 3*t3^6 - 37*t2^2*t3^6 +
↪  29*t2^4*t3^6 - 3*t2^6*t3^6 - 2*t2*t3^7 + 12*t2^3*t3^7 -
↪  10*t2^5*t3^7 + t3^8 - t2^2*t3^8 - t2^4*t3^8 + t2^6*t3^8

f99 = 15*t2^6 + 3*t2^8 + 45*t2^5*t3 - 3*t2^7*t3 +
↪  76*t2^4*t3^2 - 4*t2^6*t3^2 + 77*t2^3*t3^3 -
↪  26*t2^5*t3^3 - 23*t2^7*t3^3 + 69*t2^2*t3^4 -
↪  11*t2^4*t3^4 - 35*t2^6*t3^4 + 5*t2^8*t3^4 + 38*t2*t3^5
↪  - 41*t2^3*t3^5 - 110*t2^5*t3^5 + 17*t2^7*t3^5 + 4*t3^6
↪  - 34*t2^2*t3^6 - 54*t2^4*t3^6 + 94*t2^6*t3^6 -
↪  2*t2^8*t3^6 + 8*t2*t3^7 + 11*t2^3*t3^7 + 58*t2^5*t3^7 -
↪  25*t2^7*t3^7 - 7*t2^2*t3^8 - 7*t2^4*t3^8 - 18*t2^6*t3^8
↪  + 2*t2^8*t3^8 + 2*t2*t3^9 + t2^3*t3^9 + t2^5*t3^9 +
↪  2*t2^7*t3^9

f100 = 4*t2^6 + 38*t2^5*t3 + 8*t2^7*t3 + 2*t2^9*t3 +
↪  69*t2^4*t3^2 - 34*t2^6*t3^2 - 7*t2^8*t3^2 +
↪  77*t2^3*t3^3 - 41*t2^5*t3^3 + 11*t2^7*t3^3 + t2^9*t3^3
↪  + 76*t2^2*t3^4 - 11*t2^4*t3^4 - 54*t2^6*t3^4 -
↪  7*t2^8*t3^4 + 45*t2*t3^5 - 26*t2^3*t3^5 - 110*t2^5*t3^5
↪  + 58*t2^7*t3^5 + t2^9*t3^5 + 15*t3^6 - 4*t2^2*t3^6 -
↪  35*t2^4*t3^6 + 94*t2^6*t3^6 - 18*t2^8*t3^6 - 3*t2*t3^7
↪  - 23*t2^3*t3^7 + 17*t2^5*t3^7 - 25*t2^7*t3^7 +
↪  2*t2^9*t3^7 + 3*t3^8 + 5*t2^4*t3^8 - 2*t2^6*t3^8 +
↪  2*t2^8*t3^8

f101 = 19*t2^2 + 17*t2^4 - 3*t2^6 - t2^8 + 34*t2*t3 -
↪  16*t2^3*t3 - 46*t2^5*t3 - 4*t2^7*t3 + 19*t3^2 -
↪  50*t2^2*t3^2 - 26*t2^4*t3^2 + 22*t2^6*t3^2 +
↪  3*t2^8*t3^2 - 16*t2*t3^3 + 14*t2^3*t3^3 + 28*t2^5*t3^3
↪  + 6*t2^7*t3^3 + 17*t3^4 - 26*t2^2*t3^4 + 6*t2^4*t3^4 +
↪  6*t2^6*t3^4 - 3*t2^8*t3^4 - 46*t2*t3^5 + 28*t2^3*t3^5 +
↪  26*t2^5*t3^5 - 8*t2^7*t3^5 - 3*t3^6 + 22*t2^2*t3^6 +
↪  6*t2^4*t3^6 - 26*t2^6*t3^6 + t2^8*t3^6 - 4*t2*t3^7 +
↪  6*t2^3*t3^7 - 8*t2^5*t3^7 + 6*t2^7*t3^7 - t3^8 +
↪  3*t2^2*t3^8 - 3*t2^4*t3^8 + t2^6*t3^8

f102 = -4*t2^6 + 14*t2^5*t3 + 20*t2^7*t3 + 2*t2^9*t3 +
↪  61*t2^4*t3^2 + 26*t2^6*t3^2 - 3*t2^8*t3^2 +
↪  101*t2^3*t3^3 + 40*t2^5*t3^3 + t2^7*t3^3 - 2*t2^9*t3^3
↪  + 61*t2^2*t3^4 - 10*t2^4*t3^4 - 99*t2^6*t3^4 -
↪  20*t2^8*t3^4 + 14*t2*t3^5 + 40*t2^3*t3^5 - 45*t2^5*t3^5
↪  + 38*t2^7*t3^5 + 5*t2^9*t3^5 - 4*t3^6 + 26*t2^2*t3^6 -
↪  99*t2^4*t3^6 - 68*t2^6*t3^6 - 3*t2^8*t3^6 + 20*t2*t3^7
↪  + t2^3*t3^7 + 38*t2^5*t3^7 + 89*t2^7*t3^7 - 3*t2^2*t3^8
↪  - 20*t2^4*t3^8 - 3*t2^6*t3^8 - 30*t2^8*t3^8 + 2*t2*t3^9
↪  - 2*t2^3*t3^9 + 5*t2^5*t3^9 + 3*t2^9*t3^9

f103 = -8*t2^6 - 11*t2^5*t3 + 18*t2^7*t3 + t2^9*t3 +
↪  8*t2^4*t3^2 + 44*t2^6*t3^2 - 4*t2^8*t3^2 + 22*t2^3*t3^3
↪  + 103*t2^5*t3^3 - 16*t2^7*t3^3 - t2^9*t3^3 +
↪  8*t2^2*t3^4 + 156*t2^4*t3^4 - 156*t2^6*t3^4 -
↪  8*t2^8*t3^4 - 11*t2*t3^5 + 103*t2^3*t3^5 -
↪  298*t2^5*t3^5 + 79*t2^7*t3^5 + 3*t2^9*t3^5 - 8*t3^6 +
↪  44*t2^2*t3^6 - 156*t2^4*t3^6 + 184*t2^6*t3^6 -
↪  16*t2^8*t3^6 + 18*t2*t3^7 - 16*t2^3*t3^7 + 79*t2^5*t3^7
↪  - 46*t2^7*t3^7 + t2^9*t3^7 - 4*t2^2*t3^8 - 8*t2^4*t3^8
↪  - 16*t2^6*t3^8 + 4*t2^8*t3^8 + t2*t3^9 - t2^3*t3^9 +
↪  3*t2^5*t3^9 + t2^7*t3^9
```

```
f104 = 10*t2^6 + 6*t2^8 + 30*t2^5*t3 + 14*t2^7*t3 +
↪  61*t2^4*t3^2 + 42*t2^6*t3^2 + t2^8*t3^2 + 72*t2^3*t3^3
↪  + 40*t2^5*t3^3 - 32*t2^7*t3^3 + 75*t2^2*t3^4 +
↪  51*t2^4*t3^4 - 47*t2^6*t3^4 + t2^8*t3^4 + 44*t2*t3^5 -
↪  16*t2^3*t3^5 - 112*t2^5*t3^5 - 20*t2^7*t3^5 + 32*t3^6 +
↪  59*t2^2*t3^6 + 13*t2^4*t3^6 + 17*t2^6*t3^6 +
↪  7*t2^8*t3^6 - 8*t2*t3^7 - 152*t2^3*t3^7 - 128*t2^5*t3^7
↪  + 16*t2^7*t3^7 + 28*t3^8 + 21*t2^2*t3^8 + 109*t2^4*t3^8
↪  + 133*t2^6*t3^8 - 3*t2^8*t3^8 - 20*t2*t3^9 -
↪  32*t2^3*t3^9 - 38*t2^5*t3^9 - 42*t2^7*t3^9 + 4*t3^10 +
↪  5*t2^2*t3^10 + 6*t2^4*t3^10 + 5*t2^6*t3^10 +
↪  4*t2^8*t3^10
```

```
f105 = 32*t2^6 + 28*t2^8 + 4*t2^10 + 44*t2^5*t3 - 8*t2^7*t3
↪  - 20*t2^9*t3 + 75*t2^4*t3^2 + 59*t2^6*t3^2 +
↪  21*t2^8*t3^2 + 5*t2^10*t3^2 + 72*t2^3*t3^3 -
↪  16*t2^5*t3^3 - 152*t2^7*t3^3 - 32*t2^9*t3^3 +
↪  61*t2^2*t3^4 + 51*t2^4*t3^4 + 13*t2^6*t3^4 +
↪  109*t2^8*t3^4 + 6*t2^10*t3^4 + 30*t2*t3^5 +
↪  40*t2^3*t3^5 - 112*t2^5*t3^5 - 128*t2^7*t3^5 -
↪  38*t2^9*t3^5 + 10*t3^6 + 42*t2^2*t3^6 - 47*t2^4*t3^6 +
↪  17*t2^6*t3^6 + 133*t2^8*t3^6 + 5*t2^10*t3^6 +
↪  14*t2*t3^7 - 32*t2^3*t3^7 - 20*t2^5*t3^7 + 16*t2^7*t3^7
↪  - 42*t2^9*t3^7 + 6*t3^8 + t2^2*t3^8 + t2^4*t3^8 +
↪  7*t2^6*t3^8 - 3*t2^8*t3^8 + 4*t2^10*t3^8
```

```
f106 = -22*t2^4 - 10*t2^6 - 44*t2^3*t3 + 14*t2^5*t3 +
↪  10*t2^7*t3 - 114*t2^2*t3^2 + 15*t2^4*t3^2 +
↪  20*t2^6*t3^2 - t2^8*t3^2 - 92*t2*t3^3 + 224*t2^3*t3^3 +
↪  108*t2^5*t3^3 - 16*t2^7*t3^3 - 52*t3^4 + 121*t2^2*t3^4
↪  - 115*t2^4*t3^4 - 101*t2^6*t3^4 + 3*t2^8*t3^4 +
↪  96*t2*t3^5 + 88*t2^3*t3^5 - 12*t2^5*t3^5 + 28*t2^7*t3^5
↪  - 28*t3^6 - 63*t2^2*t3^6 - 237*t2^4*t3^6 + 27*t2^6*t3^6
↪  - 3*t2^8*t3^6 + 60*t2*t3^7 + 24*t2^3*t3^7 +
↪  148*t2^5*t3^7 - 8*t2^7*t3^7 - 41*t2^2*t3^8 -
↪  7*t2^4*t3^8 - 33*t2^6*t3^8 + t2^8*t3^8 + 12*t2^3*t3^9 -
↪  2*t2^5*t3^9 + 2*t2^7*t3^9 + t2^2*t3^10 - 2*t2^4*t3^10 +
↪  t2^6*t3^10
```

```
f107 = -52*t2^4 - 28*t2^6 - 92*t2^3*t3 + 96*t2^5*t3 +
↪  60*t2^7*t3 - 114*t2^2*t3^2 + 121*t2^4*t3^2 -
↪  63*t2^6*t3^2 - 41*t2^8*t3^2 + t2^10*t3^2 - 44*t2*t3^3 +
↪  224*t2^3*t3^3 + 88*t2^5*t3^3 + 24*t2^7*t3^3 +
↪  12*t2^9*t3^3 - 22*t3^4 + 15*t2^2*t3^4 - 115*t2^4*t3^4 -
↪  237*t2^6*t3^4 - 7*t2^8*t3^4 - 2*t2^10*t3^4 + 14*t2*t3^5
↪  + 108*t2^3*t3^5 - 12*t2^5*t3^5 + 148*t2^7*t3^5 -
↪  2*t2^9*t3^5 - 10*t3^6 + 20*t2^2*t3^6 - 101*t2^4*t3^6 +
↪  27*t2^6*t3^6 - 33*t2^8*t3^6 + t2^10*t3^6 + 10*t2*t3^7 -
↪  16*t2^3*t3^7 + 28*t2^5*t3^7 - 8*t2^7*t3^7 + 2*t2^9*t3^7
↪  - t2^2*t3^8 + 3*t2^4*t3^8 - 3*t2^6*t3^8 + t2^8*t3^8
```

```
f108 = 8*t2^6 + 37*t2^5*t3 - 6*t2^7*t3 + t2^9*t3 +
↪  57*t2^4*t3^2 - 50*t2^6*t3^2 + t2^8*t3^2 + 40*t2^3*t3^3
↪  - 31*t2^5*t3^3 + 34*t2^7*t3^3 + t2^9*t3^3 +
↪  20*t2^2*t3^4 + 89*t2^4*t3^4 - 34*t2^6*t3^4 -
↪  15*t2^8*t3^4 + 79*t2^3*t3^5 - 176*t2^5*t3^5 +
↪  35*t2^7*t3^5 + 2*t2^9*t3^5 + 37*t2^2*t3^6 -
↪  42*t2^4*t3^6 + 85*t2^6*t3^6 - 12*t2^8*t3^6 - 8*t2*t3^7
↪  + 18*t2^3*t3^7 - 67*t2^5*t3^7 - 4*t2^7*t3^7 + t2^9*t3^7
↪  - 2*t3^8 + 34*t2^2*t3^8 - 43*t2^4*t3^8 + 60*t2^6*t3^8 -
↪  5*t2^8*t3^8 + 2*t2*t3^9 - 21*t2^3*t3^9 + 13*t2^5*t3^9 -
↪  15*t2^7*t3^9 + t2^9*t3^9 + t2^2*t3^10 + 3*t2^4*t3^10 -
↪  t2^6*t3^10 + t2^8*t3^10
```

```
f109 = -2*t2^8 - 8*t2^7*t3 + 2*t2^9*t3 + 20*t2^4*t3^2 +
↪  37*t2^6*t3^2 + 34*t2^8*t3^2 + t2^10*t3^2 + 40*t2^3*t3^3
↪  + 79*t2^5*t3^3 + 18*t2^7*t3^3 - 21*t2^9*t3^3 +
↪  57*t2^2*t3^4 + 89*t2^4*t3^4 - 42*t2^6*t3^4 -
↪  43*t2^8*t3^4 + 3*t2^10*t3^4 + 37*t2*t3^5 - 31*t2^3*t3^5
↪  - 176*t2^5*t3^5 - 67*t2^7*t3^5 + 13*t2^9*t3^5 + 8*t3^6
↪  - 50*t2^2*t3^6 - 34*t2^4*t3^6 + 85*t2^6*t3^6 +
↪  60*t2^8*t3^6 - t2^10*t3^6 - 6*t2*t3^7 + 34*t2^3*t3^7 +
↪  35*t2^5*t3^7 - 4*t2^7*t3^7 - 15*t2^9*t3^7 + t2^2*t3^8 -
↪  15*t2^4*t3^8 - 12*t2^6*t3^8 - 5*t2^8*t3^8 + t2^10*t3^8
↪  + t2*t3^9 + t2^3*t3^9 + 2*t2^5*t3^9 + t2^7*t3^9 +
↪  t2^9*t3^9
```

```
f110 = 25*t2^4 + 24*t2^6 + 3*t2^8 + 50*t2^3*t3 + 22*t2^5*t3
↪  - 12*t2^7*t3 + 105*t2^2*t3^2 + 89*t2^4*t3^2 +
↪  7*t2^6*t3^2 + 3*t2^8*t3^2 + 80*t2*t3^3 - 32*t2^3*t3^3 -
↪  136*t2^5*t3^3 - 24*t2^7*t3^3 + 64*t3^4 + 89*t2^2*t3^4 +
↪  59*t2^4*t3^4 + 63*t2^6*t3^4 + 5*t2^8*t3^4 - 44*t2*t3^5
↪  - 244*t2^3*t3^5 - 184*t2^5*t3^5 - 16*t2^7*t3^5 +
↪  68*t3^6 + 91*t2^2*t3^6 + 187*t2^4*t3^6 + 141*t2^6*t3^6
↪  + t2^8*t3^6 - 104*t2*t3^7 - 160*t2^3*t3^7 -
↪  96*t2^5*t3^7 - 40*t2^7*t3^7 + 12*t3^8 + 79*t2^2*t3^8 +
↪  116*t2^4*t3^8 + 33*t2^6*t3^8 + 4*t2^8*t3^8 - 12*t2*t3^9
↪  - 30*t2^3*t3^9 - 38*t2^5*t3^9 - 4*t2^7*t3^9 +
↪  4*t2^2*t3^10 + 4*t2^4*t3^10 + 4*t2^6*t3^10
```

```
f111 = 64*t2^4 + 68*t2^6 + 12*t2^8 + 80*t2^3*t3 -
↪  44*t2^5*t3 - 104*t2^7*t3 - 12*t2^9*t3 + 105*t2^2*t3^2 +
↪  89*t2^4*t3^2 + 91*t2^6*t3^2 + 79*t2^8*t3^2 +
↪  4*t2^10*t3^2 + 50*t2*t3^3 - 32*t2^3*t3^3 -
↪  244*t2^5*t3^3 - 160*t2^7*t3^3 - 30*t2^9*t3^3 + 25*t3^4
↪  + 89*t2^2*t3^4 + 59*t2^4*t3^4 + 187*t2^6*t3^4 +
↪  116*t2^8*t3^4 + 4*t2^10*t3^4 + 22*t2*t3^5 -
↪  136*t2^3*t3^5 - 184*t2^5*t3^5 - 96*t2^7*t3^5 -
↪  38*t2^9*t3^5 + 24*t3^6 + 7*t2^2*t3^6 + 63*t2^4*t3^6 +
↪  141*t2^6*t3^6 + 33*t2^8*t3^6 + 4*t2^10*t3^6 -
↪  12*t2*t3^7 - 24*t2^3*t3^7 - 16*t2^5*t3^7 - 40*t2^7*t3^7
↪  - 4*t2^9*t3^7 + 3*t3^8 + 3*t2^2*t3^8 + 5*t2^4*t3^8 +
↪  t2^6*t3^8 + 4*t2^8*t3^8
```

```
f112 = -52*t2^4 - 28*t2^6 - 116*t2^3*t3 + 48*t2^5*t3 +
↪  52*t2^7*t3 - 150*t2^2*t3^2 + 33*t2^4*t3^2 + t2^6*t3^2 -
↪  29*t2^8*t3^2 + t2^10*t3^2 - 116*t2*t3^3 + 56*t2^3*t3^3
↪  + 248*t2^5*t3^3 + 24*t2^7*t3^3 + 12*t2^9*t3^3 - 52*t3^4
↪  + 33*t2^2*t3^4 + 208*t2^4*t3^4 - 222*t2^6*t3^4 -
↪  12*t2^8*t3^4 - 3*t2^10*t3^4 + 48*t2*t3^5 +
↪  248*t2^3*t3^5 + 40*t2^5*t3^5 + 56*t2^7*t3^5 -
↪  8*t2^9*t3^5 - 28*t3^6 + t2^2*t3^6 - 222*t2^4*t3^6 -
↪  372*t2^6*t3^6 - 6*t2^8*t3^6 + 3*t2^10*t3^6 + 52*t2*t3^7
↪  + 24*t2^3*t3^7 + 56*t2^5*t3^7 + 280*t2^7*t3^7 +
↪  4*t2^9*t3^7 - 29*t2^2*t3^8 - 12*t2^4*t3^8 - 6*t2^6*t3^8
↪  - 80*t2^8*t3^8 - t2^10*t3^8 + 12*t2^3*t3^9 -
↪  8*t2^5*t3^9 + 4*t2^7*t3^9 + 8*t2^9*t3^9 + t2^2*t3^10 -
↪  3*t2^4*t3^10 + 3*t2^6*t3^10 - t2^8*t3^10
```

```
f113 = 32*t2^6 + 28*t2^8 + 4*t2^10 + 148*t2^5*t3 +
↪  104*t2^7*t3 + 4*t2^9*t3 + 335*t2^4*t3^2 + 223*t2^6*t3^2
↪  - 11*t2^8*t3^2 - 3*t2^10*t3^2 + 428*t2^3*t3^3 +
↪  244*t2^5*t3^3 - 156*t2^7*t3^3 - 36*t2^9*t3^3 +
↪  335*t2^2*t3^4 + 260*t2^4*t3^4 - 230*t2^6*t3^4 -
↪  36*t2^8*t3^4 + 7*t2^10*t3^4 + 148*t2*t3^5 +
↪  244*t2^3*t3^5 - 356*t2^5*t3^5 - 212*t2^7*t3^5 +
↪  16*t2^9*t3^5 + 32*t3^6 + 223*t2^2*t3^6 - 230*t2^4*t3^6
↪  - 128*t2^6*t3^6 + 182*t2^8*t3^6 + t2^10*t3^6 +
↪  104*t2*t3^7 - 156*t2^3*t3^7 - 212*t2^5*t3^7 +
↪  76*t2^7*t3^7 - 36*t2^9*t3^7 + 28*t3^8 - 11*t2^2*t3^8 -
↪  36*t2^4*t3^8 + 182*t2^6*t3^8 + 76*t2^8*t3^8 +
↪  t2^10*t3^8 + 4*t2*t3^9 - 36*t2^3*t3^9 + 16*t2^5*t3^9 -
↪  36*t2^7*t3^9 - 44*t2^9*t3^9 + 4*t3^10 - 3*t2^2*t3^10 +
↪  7*t2^4*t3^10 + t2^6*t3^10 + t2^8*t3^10 + 6*t2^10*t3^10

f114 = 77*t2^6 + 24*t2^8 + 3*t2^10 + 231*t2^5*t3 -
↪  58*t2^7*t3 - 9*t2^9*t3 + 489*t2^4*t3^2 - 135*t2^6*t3^2
↪  + 13*t2^8*t3^2 - 3*t2^10*t3^2 + 593*t2^3*t3^3 -
↪  591*t2^5*t3^3 - 241*t2^7*t3^3 - 17*t2^9*t3^3 +
↪  489*t2^2*t3^4 - 667*t2^4*t3^4 + 25*t2^6*t3^4 +
↪  243*t2^8*t3^4 + 6*t2^10*t3^4 + 231*t2*t3^5 -
↪  591*t2^3*t3^5 - 311*t2^5*t3^5 + 267*t2^7*t3^5 -
↪  68*t2^9*t3^5 + 77*t3^6 - 135*t2^2*t3^6 + 25*t2^4*t3^6 +
↪  1045*t2^6*t3^6 - 146*t2^8*t3^6 + 6*t2^10*t3^6 -
↪  58*t2*t3^7 - 241*t2^3*t3^7 + 267*t2^5*t3^7 -
↪  799*t2^7*t3^7 + 31*t2^9*t3^7 + 24*t3^8 + 13*t2^2*t3^8 +
↪  243*t2^4*t3^8 - 146*t2^6*t3^8 + 277*t2^8*t3^8 -
↪  3*t2^10*t3^8 - 9*t2*t3^9 - 17*t2^3*t3^9 - 68*t2^5*t3^9
↪  + 31*t2^7*t3^9 - 45*t2^9*t3^9 + 3*t3^10 - 3*t2^2*t3^10
↪  + 6*t2^4*t3^10 + 6*t2^6*t3^10 - 3*t2^8*t3^10 +
↪  3*t2^10*t3^10

f115 = -92*t2^4 + 12*t2^8 - 172*t2^3*t3 + 220*t2^5*t3 +
↪  92*t2^7*t3 - 12*t2^9*t3 - 201*t2^2*t3^2 + 182*t2^4*t3^2
↪  + 80*t2^6*t3^2 - 126*t2^8*t3^2 + t2^10*t3^2 -
↪  172*t2*t3^3 + 168*t2^3*t3^3 + 544*t2^5*t3^3 -
↪  456*t2^7*t3^3 + 44*t2^9*t3^3 - 92*t3^4 + 182*t2^2*t3^4
↪  + 488*t2^4*t3^4 - 1332*t2^6*t3^4 + 340*t2^8*t3^4 -
↪  2*t2^10*t3^4 + 220*t2*t3^5 + 544*t2^3*t3^5 -
↪  1224*t2^5*t3^5 + 1184*t2^7*t3^5 - 84*t2^9*t3^5 +
↪  80*t2^2*t3^6 - 1332*t2^4*t3^6 + 1308*t2^6*t3^6 -
↪  508*t2^8*t3^6 + 4*t2^10*t3^6 + 92*t2*t3^7 -
↪  456*t2^3*t3^7 + 1184*t2^5*t3^7 - 792*t2^7*t3^7 +
↪  100*t2^9*t3^7 + 12*t3^8 - 126*t2^2*t3^8 + 340*t2^4*t3^8
↪  - 508*t2^6*t3^8 + 272*t2^8*t3^8 - 6*t2^10*t3^8 -
↪  12*t2*t3^9 + 44*t2^3*t3^9 - 84*t2^5*t3^9 +
↪  100*t2^7*t3^9 - 48*t2^9*t3^9 + t2^2*t3^10 -
↪  2*t2^4*t3^10 + 4*t2^6*t3^10 - 6*t2^8*t3^10 +
↪  3*t2^10*t3^10

f116 = 64*t2^4 + 68*t2^6 + 12*t2^8 + 176*t2^3*t3 +
↪  68*t2^5*t3 - 72*t2^7*t3 - 12*t2^9*t3 + 249*t2^2*t3^2 +
↪  97*t2^4*t3^2 - 9*t2^6*t3^2 + 43*t2^8*t3^2 +
↪  4*t2^10*t3^2 + 176*t2*t3^3 + 20*t2^3*t3^3 -
↪  180*t2^5*t3^3 - 100*t2^7*t3^3 - 12*t2^9*t3^3 + 64*t3^4
↪  + 97*t2^2*t3^4 + 12*t2^4*t3^4 + 190*t2^6*t3^4 +
↪  100*t2^8*t3^4 + t2^10*t3^4 + 68*t2*t3^5 - 180*t2^3*t3^5
↪  - 188*t2^5*t3^5 - 204*t2^7*t3^5 - 40*t2^9*t3^5 +
↪  68*t3^6 - 9*t2^2*t3^6 + 190*t2^4*t3^6 + 260*t2^6*t3^6 +
↪  110*t2^8*t3^6 + 5*t2^10*t3^6 - 72*t2*t3^7 -
↪  100*t2^3*t3^7 - 204*t2^5*t3^7 - 268*t2^7*t3^7 -
↪  28*t2^9*t3^7 + 12*t3^8 + 43*t2^2*t3^8 + 100*t2^4*t3^8 +
↪  110*t2^6*t3^8 + 148*t2^8*t3^8 + 3*t2^10*t3^8 -
↪  12*t2*t3^9 - 12*t2^3*t3^9 - 40*t2^5*t3^9 - 28*t2^7*t3^9
↪  - 36*t2^9*t3^9 + 4*t2^2*t3^10 + t2^4*t3^10 +
↪  5*t2^6*t3^10 + 3*t2^8*t3^10 + 3*t2^10*t3^10

f117 = -2*t2^10 - 10*t2^9*t3 + 2*t2^11*t3 + 105*t2^8*t3^2 +
↪  38*t2^10*t3^2 + t2^12*t3^2 + 480*t2^7*t3^3 -
↪  90*t2^9*t3^3 - 22*t2^11*t3^3 + 990*t2^6*t3^4 -
↪  734*t2^8*t3^4 + 8*t2^10*t3^4 + 2*t2^12*t3^4 +
↪  1248*t2^5*t3^5 - 2096*t2^7*t3^5 + 348*t2^9*t3^5 +
↪  4*t2^11*t3^5 + 990*t2^4*t3^6 - 2988*t2^6*t3^6 +
↪  1806*t2^8*t3^6 - 32*t2^10*t3^6 + 480*t2^3*t3^7 -
↪  2096*t2^5*t3^7 + 3008*t2^7*t3^7 - 804*t2^9*t3^7 -
↪  12*t2^11*t3^7 + 105*t2^2*t3^8 - 734*t2^4*t3^8 +
↪  1806*t2^6*t3^8 - 1580*t2^8*t3^8 + 195*t2^10*t3^8 +
↪  2*t2^12*t3^8 - 10*t2*t3^9 - 90*t2^3*t3^9 +
↪  348*t2^5*t3^9 - 804*t2^7*t3^9 + 444*t2^9*t3^9 -
↪  24*t2^11*t3^9 - 2*t3^10 + 38*t2^2*t3^10 + 8*t2^4*t3^10
↪  - 32*t2^6*t3^10 + 195*t2^8*t3^10 - 64*t2^10*t3^10 +
↪  t2^12*t3^10 + 2*t2*t3^11 - 22*t2^3*t3^11 + 4*t2^5*t3^11
↪  - 12*t2^7*t3^11 - 24*t2^9*t3^11 + 4*t2^11*t3^11 +
↪  t2^2*t3^12 + 2*t2^4*t3^12 + 2*t2^8*t3^12 + t2^10*t3^12

f118 = 41*t2^4 + 80*t2^6 + 34*t2^8 - 24*t2^10 - 3*t2^12 +
↪  82*t2^3*t3 + 76*t2^5*t3 - 104*t2^7*t3 - 188*t2^9*t3 +
↪  6*t2^11*t3 + 237*t2^2*t3^2 + 218*t2^4*t3^2 -
↪  218*t2^6*t3^2 - 232*t2^8*t3^2 + 181*t2^10*t3^2 +
↪  6*t2^12*t3^2 + 196*t2*t3^3 - 452*t2^3*t3^3 -
↪  1160*t2^5*t3^3 - 88*t2^7*t3^3 + 516*t2^9*t3^3 -
↪  36*t2^11*t3^3 + 92*t3^4 - 454*t2^2*t3^4 - 60*t2^4*t3^4
↪  + 1668*t2^6*t3^4 + 660*t2^8*t3^4 - 302*t2^10*t3^4 -
↪  4*t2^12*t3^4 - 332*t2*t3^5 + 152*t2^3*t3^5 +
↪  640*t2^5*t3^5 - 1000*t2^7*t3^5 - 660*t2^9*t3^5 +
↪  48*t2^11*t3^5 + 508*t2^2*t3^6 + 306*t2^4*t3^6 -
↪  576*t2^6*t3^6 + 92*t2^8*t3^6 + 244*t2^10*t3^6 +
↪  2*t2^12*t3^6 - 4*t2*t3^7 - 444*t2^3*t3^7 -
↪  248*t2^5*t3^7 + 280*t2^7*t3^7 + 188*t2^9*t3^7 -
↪  28*t2^11*t3^7 - 12*t3^8 - 18*t2^2*t3^8 + 235*t2^4*t3^8
↪  + 28*t2^6*t3^8 - 86*t2^8*t3^8 - 82*t2^10*t3^8 -
↪  t2^12*t3^8 + 12*t2*t3^9 + 22*t2^3*t3^9 - 76*t2^5*t3^9 +
↪  16*t2^7*t3^9 + 16*t2^9*t3^9 + 10*t2^11*t3^9 -
↪  t2^2*t3^10 - 4*t2^4*t3^10 + 10*t2^6*t3^10 -
↪  4*t2^8*t3^10 - t2^10*t3^10
```

```
f119 = 92*t2^4 - 12*t2^8 + 196*t2^3*t3 - 332*t2^5*t3 -
↪  4*t2^7*t3 + 12*t2^9*t3 + 237*t2^2*t3^2 - 454*t2^4*t3^2
↪  + 508*t2^6*t3^2 - 18*t2^8*t3^2 - t2^10*t3^2 +
↪  82*t2*t3^3 - 452*t2^3*t3^3 + 152*t2^5*t3^3 -
↪  444*t2^7*t3^3 + 22*t2^9*t3^3 + 41*t3^4 + 218*t2^2*t3^4
↪  - 60*t2^4*t3^4 + 306*t2^6*t3^4 + 235*t2^8*t3^4 -
↪  4*t2^10*t3^4 + 76*t2*t3^5 - 1160*t2^3*t3^5 +
↪  640*t2^5*t3^5 - 248*t2^7*t3^5 - 76*t2^9*t3^5 + 80*t3^6
↪  - 218*t2^2*t3^6 + 1668*t2^4*t3^6 - 576*t2^6*t3^6 +
↪  28*t2^8*t3^6 + 10*t2^10*t3^6 - 104*t2*t3^7 -
↪  88*t2^3*t3^7 - 1000*t2^5*t3^7 + 280*t2^7*t3^7 +
↪  16*t2^9*t3^7 + 34*t3^8 - 232*t2^2*t3^8 + 660*t2^4*t3^8
↪  + 92*t2^6*t3^8 - 86*t2^8*t3^8 - 4*t2^10*t3^8 -
↪  188*t2*t3^9 + 516*t2^3*t3^9 - 660*t2^5*t3^9 +
↪  188*t2^7*t3^9 + 16*t2^9*t3^9 - 24*t3^10 +
↪  181*t2^2*t3^10 - 302*t2^4*t3^10 + 244*t2^6*t3^10 -
↪  82*t2^8*t3^10 - t2^10*t3^10 + 6*t2*t3^11 -
↪  36*t2^3*t3^11 + 48*t2^5*t3^11 - 28*t2^7*t3^11 +
↪  10*t2^9*t3^11 - 3*t3^12 + 6*t2^2*t3^12 - 4*t2^4*t3^12 +
↪  2*t2^6*t3^12 - t2^8*t3^12

f120 = 19*t2^8 + 14*t2^10 + 3*t2^12 + 76*t2^7*t3 +
↪  32*t2^9*t3 + 4*t2^11*t3 + 244*t2^6*t3^2 + 116*t2^8*t3^2
↪  - 4*t2^10*t3^2 - 4*t2^12*t3^2 + 466*t2^5*t3^3 +
↪  34*t2^7*t3^3 - 250*t2^9*t3^3 - 26*t2^11*t3^3 +
↪  577*t2^4*t3^4 - 98*t2^6*t3^4 - 369*t2^8*t3^4 +
↪  112*t2^10*t3^4 + 6*t2^12*t3^4 + 466*t2^3*t3^5 -
↪  196*t2^5*t3^5 - 576*t2^7*t3^5 + 100*t2^9*t3^5 -
↪  18*t2^11*t3^5 + 244*t2^2*t3^6 - 98*t2^4*t3^6 -
↪  532*t2^6*t3^6 + 368*t2^8*t3^6 + 112*t2^10*t3^6 +
↪  2*t2^12*t3^6 + 76*t2*t3^7 + 34*t2^3*t3^7 -
↪  576*t2^5*t3^7 + 196*t2^7*t3^7 + 68*t2^9*t3^7 -
↪  54*t2^11*t3^7 + 19*t3^8 + 116*t2^2*t3^8 - 369*t2^4*t3^8
↪  + 368*t2^6*t3^8 + 474*t2^8*t3^8 - 122*t2^10*t3^8 +
↪  6*t2^12*t3^8 + 32*t2*t3^9 - 250*t2^3*t3^9 +
↪  100*t2^5*t3^9 + 68*t2^7*t3^9 - 516*t2^9*t3^9 +
↪  38*t2^11*t3^9 + 14*t3^10 - 4*t2^2*t3^10 +
↪  112*t2^4*t3^10 + 112*t2^6*t3^10 - 122*t2^8*t3^10 +
↪  212*t2^10*t3^10 - 4*t2^12*t3^10 + 4*t2*t3^11 -
↪  26*t2^3*t3^11 - 18*t2^5*t3^11 - 54*t2^7*t3^11 +
↪  38*t2^9*t3^11 - 40*t2^11*t3^11 + 3*t3^12 - 4*t2^2*t3^12
↪  + 6*t2^4*t3^12 + 2*t2^6*t3^12 + 6*t2^8*t3^12 -
↪  4*t2^10*t3^12 + 3*t2^12*t3^12

f121 = 69*t2^6 + 50*t2^8 + 5*t2^10 + 207*t2^5*t3 -
↪  7*t2^7*t3 - 75*t2^9*t3 - 5*t2^11*t3 + 495*t2^4*t3^2 +
↪  167*t2^6*t3^2 + 43*t2^8*t3^2 + 53*t2^10*t3^2 +
↪  2*t2^12*t3^2 + 645*t2^3*t3^3 - 109*t2^5*t3^3 -
↪  649*t2^7*t3^3 - 151*t2^9*t3^3 - 16*t2^11*t3^3 +
↪  495*t2^2*t3^4 - 202*t2^4*t3^4 - 442*t2^6*t3^4 +
↪  615*t2^8*t3^4 + 129*t2^10*t3^4 + t2^12*t3^4 +
↪  207*t2*t3^5 - 109*t2^3*t3^5 - 680*t2^5*t3^5 -
↪  60*t2^7*t3^5 - 351*t2^9*t3^5 - 47*t2^11*t3^5 + 69*t3^6
↪  + 167*t2^2*t3^6 - 442*t2^4*t3^6 + 392*t2^6*t3^6 +
↪  599*t2^8*t3^6 + 121*t2^10*t3^6 + 6*t2^12*t3^6 -
↪  7*t2*t3^7 - 649*t2^3*t3^7 - 60*t2^5*t3^7 + 84*t2^7*t3^7
↪  - 485*t2^9*t3^7 - 19*t2^11*t3^7 + 50*t3^8 +
↪  43*t2^2*t3^8 + 615*t2^4*t3^8 + 599*t2^6*t3^8 -
↪  44*t2^8*t3^8 + 180*t2^10*t3^8 + t2^12*t3^8 - 75*t2*t3^9
↪  - 151*t2^3*t3^9 - 351*t2^5*t3^9 - 485*t2^7*t3^9 -
↪  34*t2^9*t3^9 - 32*t2^11*t3^9 + 5*t3^10 + 53*t2^2*t3^10
↪  + 129*t2^4*t3^10 + 121*t2^6*t3^10 + 180*t2^8*t3^10 +
↪  14*t2^10*t3^10 + 2*t2^12*t3^10 - 5*t2*t3^11 -
↪  16*t2^3*t3^11 - 47*t2^5*t3^11 - 19*t2^7*t3^11 -
↪  32*t2^9*t3^11 - t2^11*t3^11 + 2*t2^2*t3^12 + t2^4*t3^12
↪  + 6*t2^6*t3^12 + t2^8*t3^12 + 2*t2^10*t3^12

f122 = 150*t2^6 - 44*t2^8 - 10*t2^10 + 450*t2^5*t3 -
↪  626*t2^7*t3 + 38*t2^9*t3 + 10*t2^11*t3 + 981*t2^4*t3^2
↪  - 1718*t2^6*t3^2 + 860*t2^8*t3^2 + 22*t2^10*t3^2 -
↪  t2^12*t3^2 + 1212*t2^3*t3^3 - 3962*t2^5*t3^3 +
↪  2722*t2^7*t3^3 - 462*t2^9*t3^3 - 22*t2^11*t3^3 +
↪  981*t2^2*t3^4 - 4796*t2^4*t3^4 + 6934*t2^6*t3^4 -
↪  2218*t2^8*t3^4 + 57*t2^10*t3^4 + 2*t2^12*t3^4 +
↪  450*t2*t3^5 - 3962*t2^3*t3^5 + 8072*t2^5*t3^5 -
↪  6760*t2^7*t3^5 + 902*t2^9*t3^5 + 18*t2^11*t3^5 +
↪  150*t3^6 - 1718*t2^2*t3^6 + 6934*t2^4*t3^6 -
↪  7752*t2^6*t3^6 + 3894*t2^8*t3^6 - 130*t2^10*t3^6 -
↪  2*t2^12*t3^6 - 626*t2*t3^7 + 2722*t2^3*t3^7 -
↪  6760*t2^5*t3^7 + 4920*t2^7*t3^7 - 1270*t2^9*t3^7 -
↪  10*t2^11*t3^7 - 44*t3^8 + 860*t2^2*t3^8 -
↪  2218*t2^4*t3^8 + 3894*t2^6*t3^8 - 2208*t2^8*t3^8 +
↪  194*t2^10*t3^8 + 2*t2^12*t3^8 + 38*t2*t3^9 -
↪  462*t2^3*t3^9 + 902*t2^5*t3^9 - 1270*t2^7*t3^9 +
↪  668*t2^9*t3^9 - 4*t2^11*t3^9 - 10*t3^10 + 22*t2^2*t3^10
↪  + 57*t2^4*t3^10 - 130*t2^6*t3^10 + 194*t2^8*t3^10 -
↪  116*t2^10*t3^10 - t2^12*t3^10 + 10*t2*t3^11 -
↪  22*t2^3*t3^11 + 18*t2^5*t3^11 - 10*t2^7*t3^11 -
↪  4*t2^9*t3^11 + 8*t2^11*t3^11 - t2^2*t3^12 +
↪  2*t2^4*t3^12 - 2*t2^6*t3^12 + 2*t2^8*t3^12 -
↪  t2^10*t3^12
```

```
f123 = -272*t2^6 - 104*t2^8 - 8*t2^10 - 816*t2^5*t3 +
↪  400*t2^7*t3 + 168*t2^9*t3 + 8*t2^11*t3 - 1983*t2^4*t3^2
↪  + 1060*t2^6*t3^2 - 66*t2^8*t3^2 - 68*t2^10*t3^2 +
↪  t2^12*t3^2 - 2606*t2^3*t3^3 + 4408*t2^5*t3^3 +
↪  1004*t2^7*t3^3 - 80*t2^9*t3^3 + 10*t2^11*t3^3 -
↪  1983*t2^2*t3^4 + 5968*t2^4*t3^4 - 2562*t2^6*t3^4 -
↪  2404*t2^8*t3^4 + 25*t2^10*t3^4 - 4*t2^12*t3^4 -
↪  816*t2*t3^5 + 4408*t2^3*t3^5 - 2904*t2^5*t3^5 -
↪  920*t2^7*t3^5 + 1496*t2^9*t3^5 - 16*t2^11*t3^5 -
↪  272*t3^6 + 1060*t2^2*t3^6 - 2562*t2^4*t3^6 -
↪  4088*t2^6*t3^6 + 1644*t2^8*t3^6 - 396*t2^10*t3^6 +
↪  6*t2^12*t3^6 + 400*t2*t3^7 + 1004*t2^3*t3^7 -
↪  920*t2^5*t3^7 + 6608*t2^7*t3^7 - 712*t2^9*t3^7 +
↪  52*t2^11*t3^7 - 104*t3^8 - 66*t2^2*t3^8 -
↪  2404*t2^4*t3^8 + 1644*t2^6*t3^8 - 4008*t2^8*t3^8 +
↪  142*t2^10*t3^8 - 4*t2^12*t3^8 + 168*t2*t3^9 -
↪  80*t2^3*t3^9 + 1496*t2^5*t3^9 - 712*t2^7*t3^9 +
↪  1208*t2^9*t3^9 - 16*t2^11*t3^9 - 8*t3^10 -
↪  68*t2^2*t3^10 + 25*t2^4*t3^10 - 396*t2^6*t3^10 +
↪  142*t2^8*t3^10 - 176*t2^10*t3^10 + t2^12*t3^10 +
↪  8*t2*t3^11 + 10*t2^3*t3^11 - 16*t2^5*t3^11 +
↪  52*t2^7*t3^11 - 16*t2^9*t3^11 + 10*t2^11*t3^11 +
↪  t2^2*t3^12 - 4*t2^4*t3^12 + 6*t2^6*t3^12 - 4*t2^8*t3^12
↪  + t2^10*t3^12
```