Angular Coordinates and Rational Maps

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Abstract. Associated with a triangle in the real projective plane are three standard transformations: inversion in the circumcircle, isogonal conjugation and antigonal conjugation. These are investigated in terms of angular and related coordinates, and are found to be part of a group of more general transformations. This group can also be identified with a group of automorphisms of a real two-torus. The torus is in essence the surface obtained by starting with the projective plane, performing blowups on the three vertices, and then collapsing the triangle's circumcircle and the line at infinity. Preservation of classes of triangle centers by the action of a certain discrete subgroup is also investigated. A conjecture concerning Hofstadter points is proved as an immediate consequence of this viewpoint.

Mathematics Subject Classification (2010). Primary 51N20, 14E05; Secondary 20B27, 14P25.

Keywords. Trilinear coordinates, angular coordinates, birational transformation, group action, triangle center.

1. Introduction

Advancement in the understanding of "triangle geometry" and, in particular, the interesting centers associated with a given triangle, has benefited greatly from the introduction of special coordinate systems. This is especially true of trilinear and barycentric coordinates, which are of course related to each other by means of simple scalings. Less known is the angular coordinate system associated with a triangle, though this notion has appeared in the literature for more than a century (cf. [3]).

To date, angular coordinates have only found applications in a rather limited number of circumstances. It is the intention of this article to lay out a fairly broad theory for which these coordinates are an especially natural choice, and to focus particular attention on the associated singularities. Upon resolving these singularities, a number of elegant results are easily revealed.

Some of these have already been discussed elsewhere, in other terms, but many appear to be new.

For instance, our research explores a certain two-dimensional continuous group of transformations of the plane, which is shown to include some standard involutions, namely, isogonal conjugation, antigonal conjugation and inversion in the circumcircle. A discrete subgroup of this group, containing these three involutions, has an orbit consisting almost entirely of Hofstadter points and similar points. Moreover, the elimination of the singularities that naturally occur as a result of using angular coordinates produces, as a bonus, a demonstration of Dyck's famous result on the topological equivalence of two surfaces. And so forth.

A new coordinate system is also introduced in this article. These "tricyclic" coordinates are closely related to angular coordinates, but have the advantage of being rationally related to trilinear and Cartesian coordinates. As with trilinear coordinates, it is useful to study both an exact version and a homogeneous version of the tricyclic coordinate system. Exact trilinear and tricyclic coordinates are quite different, but surprisingly, the corresponding homogeneous coordinates are reciprocals.

Section 1 of this paper introduces basic concepts, notation and conventions used throughout the paper, along with some preliminary results. Section 2 explores further the concepts of angular and tricyclic coordinates within the framework introduced in Section 1. Particular focus is placed on circles that pass through a pair of triangle vertices. Included here is a theorem that connects this framework to a generalization of a construction introduced by D. Hofstadter and C. Kimberling [6]. Section 3 looks at the three involutions mentioned earlier, and provides a rapid proof of a theorem that relates these, and which was previously proven by D. M. Bailey [2] and independently by J. Van Yzeren [10].

Section 4 identifies the singularities inherent in the usage of angular coordinates, and carefully details constructions meant to eliminate these. The angular coordinates are here treated as coordinates for a three-dimensional torus, and then restricted to a two-dimensional sub-torus. Section 5 introduces a group of birational transformations of the plane that can be identified with simple transformations of the two-dimensional torus. In fact, the latter constitute the group of isometries of the universal cover of the torus, consisting of translations and reflections. The involutions mentioned earlier correspond to three of these reflections. This group contains a certain discrete subgroup, and its action on regular and polynomial triangle centers is investigated. This is then used to prove an outstanding conjecture concerning Hofstadter points.

1.1. Notation and Conventions

Let A, B, and P be points in the plane. Define the *directed angle* $\measuredangle APB$ to be the angle through which the line $\overleftrightarrow{AP}$ can be rotated about P to coincide with the line $\overleftrightarrow{BP}$. The angle is signed, with positive values indicating counterclockwise rotation, and is only well-defined modulo π. Any equation involving directed angles should be considered modulo π.

We record some immediately observed properties of directed angles below:

Lemma 1.1. *Let A, B, C, and P be points in the plane. Then*
(i) $\measuredangle APB = -\measuredangle BPA$,
(ii) $\measuredangle BAC + \measuredangle CBA + \measuredangle ACB = 0$,
(iii) $\measuredangle APB + \measuredangle BPC + \measuredangle CPA = 0$.

The inscribed angle theorem can be written in terms of directed angles as follows:

Lemma 1.2. *Let A, B, P, and Q be points in the plane. Then A, B, P, and Q are concyclic if and only if $\measuredangle APB = \measuredangle AQB$ if and only if $\measuredangle PAQ = \measuredangle PBQ$.*

Proof. The key difference from the traditional inscribed angle theorem can be seen as follows: If P and Q are on opposite sides of a chord AB of a circle, then $\angle APB = \pi - \angle AQB$. But the directed angles $\measuredangle APB$ and $\measuredangle AQB$ must have opposite orientation in this case, so $\measuredangle APB = \pi + \measuredangle AQB = \measuredangle AQB$. □

We will fix a triangle ΔABC with circumcenter O and circumradius R and with A, B, and C not collinear. The interior angles at A, B, and C will be denoted by θ_1, θ_2, and θ_3, respectively. We will write $L_i = R\sin\theta_i$ and $M_i = R\cos\theta_i$. Observe that L_i is half the length of its corresponding edge and M_i is the signed distance from O to the corresponding edge. This is shown in Fig. 1.

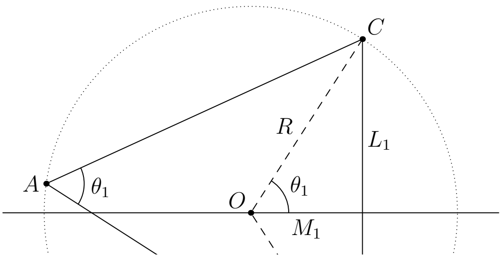


FIGURE 1. The quantities L_1 and M_1. For this triangle, $M_1 > 0$.

A subscript used to indicate an edge or vertex of ΔABC may be dropped when it can be understood from context. Subscripts may also be dropped in expressions that would be written the same way for each subscript. For example, $L = R\sin\theta$ means that $L_i = R\sin\theta_i$ for each i.

1.2. Bailey Circles

Circles which pass through two vertices of ΔABC will be referred to repeatedly in what follows, so it will be convenient to use non-standard terminology and refer to any such circle as a *Bailey circle*[1]. To be specific, we will say that a circle through A and B is a Bailey circle for the edge AB, and similarly for the other edges. The sidelines of ΔABC will be considered Bailey circles with infinite radii.

A Bailey circle for an edge E of ΔABC can be uniquely identified by the location of its center along the perpendicular bisector $E^\perp$ of E. A coordinate may be assigned to any point on $E^\perp$ by specifying its signed distance from the circumcenter of ΔABC, with the positive direction coinciding with the outward-pointing normal to E. In this way, any configuration of Bailey circles, one for each edge of ΔABC, may be specified as a triple (c_1, c_2, c_3): Let X, Y, and Z be the centers of the Bailey circles for edges BC, CA, and AB, respectively. Then let c_1, c_2, and c_3 be the coordinates, as described above, of X, Y, and Z, respectively.

Another method of specifying Bailey circles uses directed angles. Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be Bailey circles for edges BC, CA, and AB, respectively. Choose any point $P_1 \neq B, C$ on $\mathcal{C}_1$, $P_2 \neq C, A$ on $\mathcal{C}_2$, and $P_3 \neq A, B$ on $\mathcal{C}_3$. Then define

$$\psi_1 = \measuredangle BP_1C, \quad \psi_2 = \measuredangle CP_2A, \quad \psi_3 = \measuredangle AP_3B. \tag{1.1}$$

By Lemma 1.2, these values are well-defined and uniquely specify each circle. An individual Bailey circle can be specified by the value in Eq. (1.1) corresponding to its edge, or a configuration of Bailey circles can be specified as a triple (ψ_1, ψ_2, ψ_3).

We now establish the relationship between c and ψ in Lemma 1.3 and Lemma 1.4.

Lemma 1.3. $\cot(\psi) = \dfrac{M - c}{L}$ *and* $\cot(\theta - \psi) = \dfrac{M - R^2\, c^{-1}}{L}$.

Proof. We prove the first assertion for the edge $E = BC$. Let $\mathcal{C}$ be a Bailey circle for BC with center X. Let P and Q be the two points at which $\mathcal{C}$ meets $E^\perp$, chosen so that P and X are on the same side of BC. This is shown in Fig. 2.

By the inscribed angle theorem, $\angle BPC = \frac{1}{2}\angle BXC = \angle QXC$. Observe that the rotation of $\overleftrightarrow{PB}$ onto $\overleftrightarrow{PC}$ is a clockwise acute angle when $c > M$ and a counterclockwise acute angle when $c < M$, and similarly for the rotation of $\overleftrightarrow{XQ}$ onto $\overleftrightarrow{XC}$. Since $\angle BPC$ and $\angle QXC$ are oriented the same way,

$$\psi = \measuredangle BPC = \measuredangle QXC.$$

Now let $\sigma = \operatorname{sgn}(M - c)$ so that $\measuredangle QXC = \sigma \angle QXC$ and the absolute distance from X to BC is $\sigma(M - c)$. Then

$$\cot \measuredangle QXC = \sigma \cot \angle QXC = \sigma \frac{\sigma(M - c)}{L} = \frac{M - c}{L}.$$

[1]in recognition of D. M. Bailey's investigations into these circles (cf. [2])

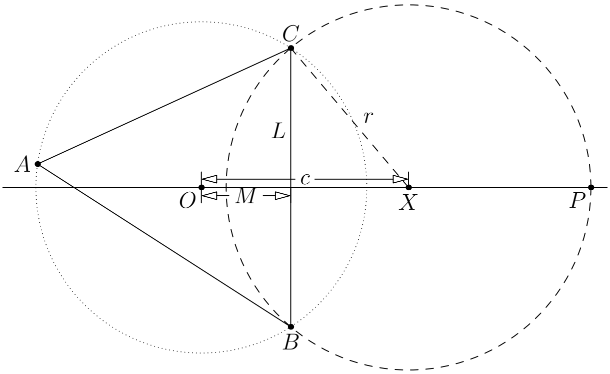


FIGURE 2. A Bailey circle with radius r.

For the other equality,

$$\cot(\theta-\psi)=\frac{\cot(\theta)\cot(\psi)+1}{\cot(\psi)-\cot(\theta)}\cdot\frac{L^2}{L^2}=\frac{M(M-c)+L^2}{L(M-c)-LM}=\frac{R^2-Mc}{-Lc}.$$

The last equality follows from the fact that $M^2+L^2=R^2$. □

Lemma 1.4. $c=R\dfrac{\sin(\psi-\theta)}{\sin(\psi)}$.

Proof. Replacing $L=R\sin\theta$ and $M=R\cos\theta$ in Lemma 1.3 yields

$$\begin{aligned}c&=R\cos(\theta)-R\sin(\theta)\cot(\psi)\\&=R\frac{\cos(\theta)\sin(\psi)-\sin(\theta)\cos(\psi)}{\sin(\psi)}\end{aligned}$$

The result follows. □

Finally, we give a condition for neighboring Bailey circles to be tangent.

Lemma 1.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Bailey circles for edges BC and CA, and let ψ_1 and ψ_2 be their respective ψ-coordinates. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent if and only if $\psi_1+\psi_2=-\theta_3$.*

Proof. From Fig. 2, one can deduce that

$$\begin{aligned}\measuredangle OCX&=\measuredangle OCB+\measuredangle BCX\\&=(\pi/2-\theta_1)+(\pi/2+\psi_1)=\psi_1-\theta_1.\end{aligned}$$

By a symmetrical argument, $\measuredangle OCY=-(\psi_2-\theta_2)$. Therefore

$$\begin{aligned}\measuredangle YCX&=\measuredangle YCO+\measuredangle OCX\\&=(\psi_2-\theta_2)+(\psi_1-\theta_1)=\psi_1+\psi_2+\theta_3.\end{aligned}$$

But $\mathcal{C}_1$ and $\mathcal{C}_2$ are tangent precisely when $\measuredangle YCX=0$. □

1.3. Angular and Tricyclic Coordinates

Observe that if $P \neq A, B, C$, then there is exactly one configuration of Bailey circles such that each Bailey circle passes through P.

This suggests defining coordinates on the plane by specifying the configuration of Bailey circles induced by each point. Such coordinates are particularly useful when considering those transformations of the plane which send each Bailey circle to another—this includes antigonal conjugation, isogonal conjugation, and inversion in the circumcircle, as we will see in Section 3.

Let $P \neq A, B, C$ be the common intersection of a configuration of Bailey circles. Setting $P_1 = P_2 = P_3 = P$ in Eq. (1.1) yields a triple (ψ_1, ψ_2, ψ_3) called the *angular coordinates of* P. In [3, Chapter II] and [9], angular coordinates are defined similarly, but only for points inside ΔABC, and using the absolute angles $\angle BPC$, $\angle CPA$, and $\angle APB$.

Another possible coordinate system uses the triple (c_1, c_2, c_3), defined above, to specify the configuration of Bailey circles. It will be convenient to use non-standard terminology and refer to (c_1, c_2, c_3) as the *exact tricyclic coordinates of* P. For any $\lambda \in \mathbb{R}^*$, we will refer to $(\lambda c_1 : \lambda c_2 : \lambda c_3)$ as the *homogeneous tricyclic coordinates of* P.

As we will see in Eq. (2.7) and Eq. (2.8), tricyclic coordinates are rationally related to trilinear, barycentric, and Cartesian coordinates. The relationship between angular and tricyclic coordinates is described by Lemma 1.3 and Lemma 1.4.

Remark 1.6. Note that

(i) Points not on the circumcircle have unique, well-defined angular and tricyclic coordinates, with the caveat that points on the sidelines have one infinite tricyclic coordinate.

(ii) Points on the circumcircle other than A, B, and C cannot be distinguished using angular or tricyclic coordinates.

(iii) If $P = A$, B, or C, then there are infinitely many configurations of Bailey circles such that each Bailey circle passes through P. Hence P does not have well-defined angular or tricyclic coordinates.

These issues are analyzed in detail in Section 4.

2. Properties of Angular and Tricyclic Coordinates

2.1. Exactness

Proposition 2.1. *If* (ψ_1, ψ_2, ψ_3) *are the angular coordinates of a point* P*, then*

$$\psi_1 + \psi_2 + \psi_3 = 0. \tag{2.1}$$

Proof. By definition,

$$\psi_1 + \psi_2 + \psi_3 = \measuredangle BPC + \measuredangle CPA + \measuredangle APB.$$

The right-hand side is 0 by Lemma 1.1. □

The following provides a partial converse.

Proposition 2.2. *Let ψ_1, ψ_2, and ψ_3 be any triple of directed angles such that*

$$\psi_1 + \psi_2 + \psi_3 = 0.$$

Consider the configuration of Bailey circles given by (ψ_1, ψ_2, ψ_3). Exactly one of the following is true:

(i) *All three Bailey circles are sidelines ($\psi_1 = \psi_2 = \psi_3 = 0$),*
(ii) *At least one Bailey circle is the circumcircle and the other two are tangent,*
(iii) *There is a common point of intersection P not on the circumcircle.*

The case that all three Bailey circles are the circumcircle ($\psi_1 = \theta_1, \psi_2 = \theta_2, \psi_3 = \theta_3$) is a special case of (ii).

Proof. Clearly (i), (ii), and (iii) are each possible and are mutually exclusive. We will show they are exhaustive. Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ denote the Bailey circles in the configuration for the edges BC, CA, and AB, respectively.

Suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at a point P not on the circumcircle. Let $\mathcal{K}$ be the circle APB and $\mathring{\psi}$ its ψ-coordinate. Then the angular coordinates of P are $(\psi_1, \psi_2, \mathring{\psi})$, so by Proposition 2.1, $\psi_1 + \psi_2 + \mathring{\psi} = 0$. Hence $\psi_3 = \mathring{\psi}$ and $\mathcal{C}_3 = \mathcal{K}$. This is case (iii).

If $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at a point on the circumcircle other than C, then at least one of $\mathcal{C}_1$ and $\mathcal{C}_2$ must be the circumcircle. Given the condition that $\psi_1 + \psi_2 + \psi_3 = 0$, Lemma 1.5 implies that two Bailey circles are tangent if and only if the other is the circumcircle. So this is case (ii).

Finally, if $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect only at C, then they are either tangent, which is case (ii), or they are both sidelines. If they are both sidelines, then $\psi_1 = \psi_2 = 0$. This implies $\psi_3 = 0$, which is case (i). □

Proposition 2.3. *If (c_1, c_2, c_3) are the exact tricyclic coordinates of a point P, then*

$$R(L_1c_1 + L_2c_2 + L_3c_3) = L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2. \tag{2.2}$$

Proof. It is sufficient to show that Eq. (2.2) is equivalent to Eq. (2.1).

Let $\Theta = e^{i\theta}$, so that $R\Theta = M + iL$. Then

$$e^{2i\psi} = \frac{\cot\psi + i}{\cot\psi - i} = \frac{(M - c) + Li}{(M - c) - Li} = \frac{R\Theta - c}{R\overline{\Theta} - c}, \tag{2.3}$$

where the middle equality follows from Lemma 1.3. Now let

$$\xi = (R\Theta_1 - c_1)(R\Theta_2 - c_2)(R\Theta_3 - c_3).$$

Then

$$e^{2i(\psi_1+\psi_2+\psi_3)} = \frac{(R\Theta_1 - c_1)(R\Theta_2 - c_2)(R\Theta_3 - c_3)}{(R\overline{\Theta_1} - c_1)(R\overline{\Theta_2} - c_2)(R\overline{\Theta_3} - c_3)} = \xi/\overline{\xi}.$$

Therefore Eq. (2.1) holds if and only if $\xi/\overline{\xi} = 1$, or equivalently $\operatorname{Im}(\xi) = 0$.

Observe that $\Theta_1\Theta_2\Theta_3 = e^{i(\theta_1+\theta_2+\theta_3)} = -1$, so $\operatorname{Im}(\Theta_1\Theta_2\Theta_3) = 0$. Also, $\Theta_1\Theta_2 = -\overline{\Theta_3}$, so $\operatorname{Im}(\Theta_1\Theta_2) = \operatorname{Im}(\Theta_3) = L_3$; similarly, $\operatorname{Im}(\Theta_1\Theta_3) = L_2$ and $\operatorname{Im}(\Theta_2\Theta_3) = L_1$. Expanding ξ and extracting the imaginary part yields

$$\operatorname{Im}(\xi) = R(L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2) - R^2(L_1c_1 + L_2c_2 + L_3c_3).$$

Setting this equal to zero and rearranging yields Eq. (2.2). □

Corollary 2.4. *If P is not on the circumcircle and has homogeneous tricyclic coordinates $(c_1 : c_2 : c_3)$, then it has exact tricyclic coordinates (Kc_1, Kc_2, Kc_3), where*

$$K = R\frac{L_1c_1 + L_2c_2 + L_3c_3}{L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2}. \tag{2.4}$$

2.2. Construction from Tricyclic Coordinates

Let P be a point not on the circumcircle or sidelines of ΔABC. Let X denote the center of the circle BPC. The point X can be obtained by intersecting the perpendicular bisectors of the chords PB and PC, and the exact tricyclic coordinate c_1 is equal to the signed length of OX. The other tricyclic coordinates c_2 and c_3 can be obtained by constructing the center Y of the circle CPA and the center Z of the circle APB, respectively, in the same way.

It follows that the triangle ΔXYZ formed by the three Bailey circle centers is the antipedal triangle of ΔABC with respect to P, scaled by $\frac{1}{2}$. This construction is shown in Fig. 3.

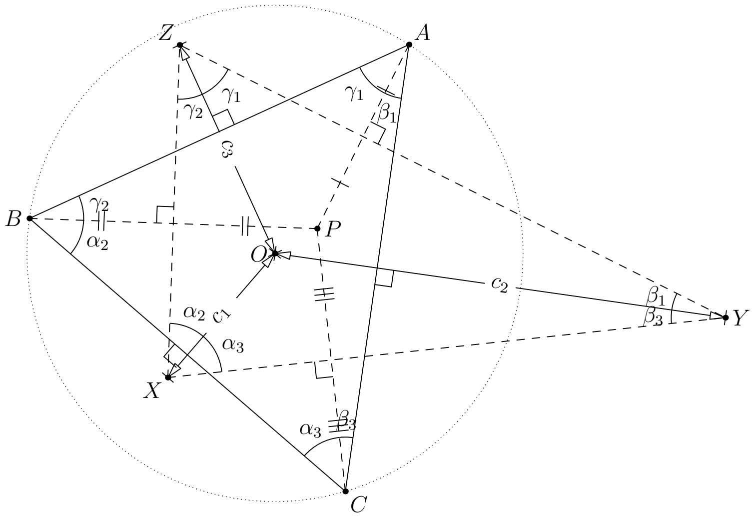


FIGURE 3. ΔXYZ is half the antipedal triangle of ΔABC with respect to P.

Observe that in Fig. 3, the triangle ΔXYZ can be scaled relative to O without changing P. Thus a point P can be constructed directly from its homogeneous tricyclic coordinates. This is stated in terms of orthologic triangles in Proposition 2.5.

Recall that triangles ΔABC and ΔXYZ are said to be orthologic if there exists a point P such that $\overleftrightarrow{PA}$ is perpendicular to $\overleftrightarrow{YZ}$, $\overleftrightarrow{PB}$ is perpendicular to $\overleftrightarrow{ZX}$, and $\overleftrightarrow{PC}$ is perpendicular to $\overleftrightarrow{XY}$. In this case P is called the orthology center of ΔABC with respect to ΔXYZ. This configuration is symmetric in the sense that, by a well-known classical result, there must exist a point Q which is the orthology center of ΔXYZ with respect to ΔABC.

Proposition 2.5. *Let c_1, c_2, and c_3 be any triple of non-zero real numbers.*

Let $\mathring{X}$, $\mathring{Y}$, and $\mathring{Z}$ have signed distances from O equal to c_1, c_2, and c_3, respectively, in the directions perpendicular to BC, CA, and AB, respectively, so that O is the orthology center of $\Delta\mathring{X}\mathring{Y}\mathring{Z}$ with respect to ΔABC.

Let P denote the orthology center of ΔABC with respect to $\Delta\mathring{X}\mathring{Y}\mathring{Z}$. Then P has homogeneous tricyclic coordinates $(c_1 : c_2 : c_3)$.

Proof. Let ΔXYZ be the antipedal triangle of ΔABC with respect to P, scaled by $\frac{1}{2}$, as in Fig. 3. Then ΔABC and ΔXYZ are orthologic with orthology centers P and O, respectively. Since ΔABC has orthology center P with respect to both ΔXYZ and $\Delta\mathring{X}\mathring{Y}\mathring{Z}$, the latter two triangles must both have sidelines perpendicular to $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, and $\overleftrightarrow{CP}$; hence, they are homothetic. Since they both have orthology center O with respect to ΔABC, they must in fact be homothetic with respect to O. So ΔXYZ is equal to $\Delta\mathring{X}\mathring{Y}\mathring{Z}$ scaled by some factor K with respect to O. But X, Y, and Z are the centers of the circles BPC, CPA, and APB, respectively, so the exact tricyclic coordinates of P are (Kc_1, Kc_2, Kc_3). □

2.3. Trilinear and Cartesian Coordinates

Proposition 2.6. *Let P be a point not on the sidelines of ΔABC. Let (ℓ_1, ℓ_2, ℓ_3) and (c_1, c_2, c_3) denote the exact trilinear and exact tricyclic coordinates, respectively, of P. Then for each i,*

$$2c_i\ell_i = -\mathcal{P}, \tag{2.5}$$

where $\mathcal{P} = |OP|^2 - R^2$ is the power of P for the circumcircle, and

$$c_i\ell_i = \frac{|\Delta ABC|}{L_1c_1^{-1} + L_2c_2^{-1} + L_3c_3^{-1}}. \tag{2.6}$$

Proof. If P is on the circumcircle, then Eq. (2.5) is clearly true. Now assume P is not on the circumcircle, and let ΔXYZ be the triangle formed by the centers of the Bailey circles induced by P, as in Fig. 3. Observe that $\overrightarrow{PB} \cdot \overrightarrow{OX} = \ell_1c_1$. Also, since $\overrightarrow{PB} \cdot (\overrightarrow{XP} + \overrightarrow{PB}/2) = 0$, it follows that $2\overrightarrow{PB} \cdot \overrightarrow{PX} = |PB|^2$. Therefore

$$\begin{aligned} 2c_1\ell_1 &= 2\overrightarrow{PB} \cdot (\overrightarrow{OP} + \overrightarrow{PX}) \\ &= 2\overrightarrow{PB} \cdot \overrightarrow{OP} + |PB|^2 \\ &= |\overrightarrow{OP} + \overrightarrow{PB}|^2 - |OP|^2 \\ &= R^2 - |OP|^2. \end{aligned}$$

The arguments for $c_2\ell_2$ and $c_3\ell_3$ are similar, so Eq. (2.5) holds.

Since

$$L_1\ell_1 + L_2\ell_2 + L_3\ell_3 = |\Delta ABC|,$$

it follows that

$$-\mathcal{P}L_1c_1^{-1} - \mathcal{P}L_2c_2^{-1} - \mathcal{P}L_3c_3^{-1} = 2|\Delta ABC|.$$

Solving for $\mathcal{P}$ yields Eq. (2.6). □

Applying Lemma 1.4, Eq. (2.5) can be rewritten as

$$2R\ell = -\mathcal{P}\frac{\sin(\psi)}{\sin(\psi - \theta)}.$$

This is proved in [3, Chapter II], which discusses the notion of "power," although the angular coordinates in [3] differ slightly from those defined here.

Corollary 2.7. *Let P be a point not on the circumcircle or sidelines of ΔABC. Let $(\ell_1 : \ell_2 : \ell_3)$ and $(c_1 : c_2 : c_3)$ denote the homogeneous trilinear and homogeneous tricyclic coordinates, respectively, of P. Then*

$$(\ell_1 : \ell_2 : \ell_3) = (c_1^{-1} : c_2^{-1} : c_3^{-1}) \tag{2.7}$$

Proof. This follows directly from Proposition 2.6. □

Corollary 2.8. *Let P be a point not on the circumcircle or sidelines of ΔABC. Suppose that the Cartesian coordinates of P are given by dehomogenizing $(x : y : z) \in \mathbb{RP}^2$ at $z = 1$, and that the Cartesian coordinates of A, B, and C are given by (x_1, y_1), (x_2, y_2), and (x_3, y_3), respectively. Let $(c_1 : c_2 : c_3)$ denote the homogeneous tricyclic coordinates of P. Then*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\, c_1^{-1} \\ L_2\, c_2^{-1} \\ L_3\, c_3^{-1} \end{pmatrix} \tag{2.8}$$

Proof. From Eq. (2.7), the barycentric coordinates of P are

$$(L_1\, c_1^{-1} : L_2\, c_2^{-1} : L_3\, c_3^{-1}).$$

The result follows. □

Remark 2.9. Both Corollary 2.7 and Corollary 2.8 can be extended to the case that P is on a sideline of ΔABC by considering each tricyclic coordinate as belonging to $\mathbb{RP}^1$ and scaling the homogeneous quantities in the typical way. This is discussed further in Section 4.

Proposition 2.10. *Suppose a plane transformation is given in exact tricyclic coordinates as*

$$c \mapsto c' = \frac{\alpha c + \beta}{\gamma c + \delta}$$

for some α_i, β_i, γ_i, and δ_i. Then it can be written in homogeneous trilinear coordinates as

$$\ell \mapsto \ell' = \frac{\gamma R\Lambda_2 + \delta\Lambda_1\ell}{\alpha R\Lambda_2 + \beta\Lambda_1\ell}$$

where $\Lambda_1 = L_1\ell_1 + L_2\ell_2 + L_3\ell_3$ and $\Lambda_2 = L_1\ell_2\ell_3 + L_2\ell_1\ell_3 + L_3\ell_1\ell_2$.

Proof. Let $K_1 = L_1c_1 + L_2c_2 + L_3c_3$ and $K_2 = L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2$.

The original map is in terms of exact tricyclic coordinates; to express the map in terms of homogeneous tricyclic coordinates, each c value must be scaled by $RK_1K_2^{-1}$, as in Eq. (2.4):

$$c' = \frac{\alpha RK_1c + \beta K_2}{\gamma RK_1c + \delta K_2}.$$

Next we replace $c = -\mathcal{P}/(2\ell)$ and $c' = -\mathcal{P}/(2\ell')$, as in Eq. (2.7). This transforms K_1 into $-\mathcal{P}\Lambda_2/(2\ell_1\ell_2\ell_3)$ and K_2 into $\mathcal{P}^2\Lambda_1/(4\ell_1\ell_2\ell_3)$. Thus

$$\frac{-\mathcal{P}}{2\ell'} = \frac{\alpha R\ell^{-1}\Lambda_2 + \beta\Lambda_1}{\gamma R\ell^{-1}\Lambda_2 + \delta\Lambda_1},$$

where we have cancelled a factor of $\mathcal{P}^2/(4\ell_1\ell_2\ell_3)$ from the numerator and denominator. Finally, the factor of $-\mathcal{P}/2$ on the left can be dropped since these are homogeneous coordinates. Rearranging completes the proof. □

2.4. Construction from Angular Coordinates

Here we describe a construction very similar to the one used in [6] to define Hofstadter points. It gives a direct geometric construction of a point from its angular coordinates, and in fact yields a point for almost any triple of directed angles.

Theorem 2.11. *Let ψ_1, ψ_2, and ψ_3 be any triple of directed angles such that $\psi_i \neq 0, \theta_i$. Let A', B', and C' be the points satisfying*

$$\measuredangle BAC' = \measuredangle B'AC = \psi_1,$$
$$\measuredangle CBA' = \measuredangle C'BA = \psi_2,$$
$$\measuredangle ACB' = \measuredangle A'CB = \psi_3,$$

as in Fig. 4. Then $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are concurrent, meeting in a point P, and

(i) *P has homogeneous trilinear coordinates*

$$\ell = \frac{\sin(\psi)}{\sin(\psi - \theta)},$$

(ii) *P has homogeneous tricyclic coordinates*

$$c = R\frac{\sin(\psi - \theta)}{\sin(\psi)},$$

(iii) *If $\psi_1 + \psi_2 + \psi_3 = 0$, then the tricyclic coordinates in (ii) are exact and the angular coordinates of P are (ψ_1, ψ_2, ψ_3).*

Proof. To establish (i), we follow the same reasoning as in [6]. First, note that a given line through A, B, or C includes all points with some fixed ratio of trilinear coordinates $[\ell_2 : \ell_3]$, $[\ell_1 : \ell_3]$, or $[\ell_1 : \ell_2]$, respectively. From Fig. 4, points on $\overleftrightarrow{CA'}$ satisfy

$$[\ell_1 : \ell_2] = [\sin(\psi_3) : \sin(\theta_3 - \psi_3)]$$

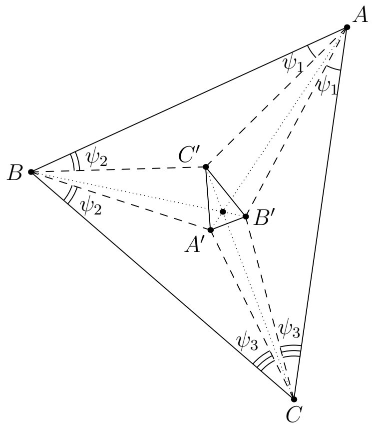


FIGURE 4. P is the intersection of $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$.

and points on $\overleftrightarrow{BA'}$ satisfy

$$[\ell_1 : \ell_3] = [\,\sin(\psi_2) : \sin(\theta_2 - \psi_2)\,].$$

It follows that A', and consequently other points on $\overleftrightarrow{AA'}$, satisfies

$$[\ell_2 : \ell_3] = [\,\sin(\psi_2)\sin(\theta_3 - \psi_3) : \sin(\theta_2 - \psi_2)\sin(\psi_3)\,]. \tag{2.9}$$

Analogous reasoning shows that $\overleftrightarrow{BB'}$ is given by

$$[\ell_1 : \ell_3] = [\,\sin(\psi_1)\sin(\theta_3 - \psi_3) : \sin(\theta_1 - \psi_1)\sin(\psi_3)\,] \tag{2.10}$$

and $\overleftrightarrow{CC'}$ is given by

$$[\ell_1 : \ell_2] = [\,\sin(\psi_1)\sin(\theta_2 - \psi_2) : \sin(\theta_1 - \psi_1)\sin(\psi_2)\,]. \tag{2.11}$$

The point P with trilinear coordinates

$$\left(\frac{\sin(\psi_1)}{\sin(\psi_1 - \theta_1)} : \frac{\sin(\psi_2)}{\sin(\psi_2 - \theta_2)} : \frac{\sin(\psi_3)}{\sin(\psi_3 - \theta_3)}\right)$$

satisfies each of Eq. (2.9), Eq. (2.10), and Eq. (2.11), so it must be the common intersection of $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$.

Part (ii) follows directly from (i) by Corollary 2.7.

Now suppose that $\psi_1 + \psi_2 + \psi_3 = 0$. Consider the configuration of Bailey circles given by (c_1, c_2, c_3), with c_i as in (ii). By Lemma 1.4, the same configuration is given by (ψ_1, ψ_2, ψ_3). By Proposition 2.2, these three circles must have some common point of intersection Q which is not on the circumcircle. Therefore Q has angular coordinates (ψ_1, ψ_2, ψ_3) and exact tricyclic coordinates (c_1, c_2, c_3). By (ii) and the uniqueness of tricyclic coordinates for points not on the circumcircle, $Q = P$. □

Remark 2.12. In the case that $\psi_2 = -\psi_3$, the lines which would intersect to form A' are parallel. In this case the expression Eq. (2.9) gives the line through A parallel to both of these, and the proof continues with this line in place of $\overleftrightarrow{AA'}$. The same principle holds for B' and C'.

Remark 2.13. In the case that $\psi = r\theta$ and $r \neq 0, 1$, this construction yields the Hofstadter r-point, as defined in [6]. This will be explored further in Section 5.4.

If $\psi_1 = \psi_2 = \psi_3 = -\pi/3$, then P is the first isogonic center. If $\psi_1 = \psi_2 = \psi_3 = \pi/3$, then P is the second isogonic center. By Theorem 2.11, it follows that the angular coordinates of the first and second isogonic centers are $(-\pi/3, -\pi/3, -\pi/3)$ and $(\pi/3, \pi/3, \pi/3)$, respectively.

If $\psi = \theta/2$, then P is the incenter I. It does not follow that the angular coordinates of I are $\psi = \theta/2$, because in this case $\psi_1 + \psi_2 + \psi_3 \neq 0$. Indeed, it is straightforward to deduce that the angular coordinates of I are in fact $\psi = (\theta+\pi)/2$ (and therefore repeating the construction using these angles still produces the incenter I). However, the values of ψ used in the construction are related to the ψ-coordinates of Bailey circles, as described in Corollary 2.14.

Corollary 2.14. *Let $\mathring{X}$, $\mathring{Y}$, and $\mathring{Z}$ denote the centers of the Bailey circles in the configuration given by (ψ_1, ψ_2, ψ_3). Let X, Y, and Z denote the centers of the circles BPC, CPA, and APB, where P is constructed as in Theorem 2.11 with angles ψ_1, ψ_2, and ψ_3. Then*

(i) *ΔXYZ and $\Delta\mathring{X}\mathring{Y}\mathring{Z}$ are homothetic with respect to O.*
(ii) *If $\psi_1 + \psi_2 + \psi_3 = 0$, then $\Delta XYZ = \Delta\mathring{X}\mathring{Y}\mathring{Z}$.*

Proof. By Lemma 1.4, $\Delta\mathring{X}\mathring{Y}\mathring{Z}$ is formed by the centers of the Bailey circles given by the homogeneous tricyclic coordinates of P. Therefore ΔXYZ, the configuration given by the exact tricyclic coordinates of P, is obtained from $\Delta\mathring{X}\mathring{Y}\mathring{Z}$ by scaling by some factor K with respect to O. □

3. Transformations

Given a triangle ΔABC, we will say that a birational automorphism of the plane F preserves Bailey circles if, for any Bailey circle $\mathcal{C}$ for an edge of ΔABC, there is a Bailey circle $\mathcal{C}'$ for the same edge such that F restricts to a map $\mathcal{C} \dashrightarrow \mathcal{C}'$.

Each tricyclic coordinate of a point P specifies a Bailey circle through P. Hence, a map F preserves Bailey circles if and only if it is "diagonal" when written in tricyclic coordinates, in the sense that F can be expressed as

$$(c_1, c_2, c_3) \mapsto (f_1(c_1), f_2(c_2), f_3(c_3)) \tag{3.1}$$

for some functions f_i. The same principle holds for angular coordinates.

We will show that antigonal conjugation, isogonal conjugation, and inversion in the circumcircle can each be expressed in the form of Eq. (3.1). The formulas in terms of angular coordinates are equivalent to the "characteristic equations" appearing in [10]. In what follows, we fix a generic edge E of ΔABC and consider each coordinate independently of the others.

3.1. Antigonal Conjugation

What we refer to here as antigonal conjugates are referred to as reflective points in [2], reflective conjugates in [4], and antigonal pairs in [10].

Let P be a point in the plane other than A, B, and C, and consider the three Bailey circles induced by P. Reflect each circle over its corresponding edge. The reflected circles will intersect in a unique point P' called the antigonal conjugate of P.

Proposition 3.1. *Antigonal conjugation is given in angular and exact tricyclic coordinates, respectively, by*

$$(\psi_1, \psi_2, \psi_3) \mapsto (-\psi_1, -\psi_2, -\psi_3)$$

and

$$(c_1, c_2, c_3) \mapsto (2M_1 - c_1, 2M_2 - c_2, 2M_3 - c_3)$$

Proof. Let $\mathcal{C}$ be a Bailey circle for E and let $\mathcal{C}'$ be its image under antigonal conjugation. Let c and c' be the coordinates of $\mathcal{C}$ and $\mathcal{C}'$, respectively. Since the midpoint of E is exactly between the centers of $\mathcal{C}$ and $\mathcal{C}'$, $\dfrac{c + c'}{2} = M$. Therefore $c' = 2M - c$.

Let Q denote one of the two points where $E^{\perp}$ and $\mathcal{C}$ intersect. Then if Q' is the reflection of Q over E, it must sit on $\mathcal{C}'$. That is, $\psi' = \measuredangle AQ'B$ is the same as $\psi = \measuredangle AQB$, but with opposite orientation. Hence $\psi' = -\psi$. □

3.2. Isogonal Conjugation

Isogonal conjugation is defined as follows: Let P be a point in the plane. Reflect the lines $\overleftrightarrow{AP}$, $\overleftrightarrow{BP}$, and $\overleftrightarrow{CP}$ over the internal angle bisectors at A, B, and C, respectively. The reflected lines will intersect in a point P', called the isogonal conjugate of P.

Proposition 3.2. *Isogonal conjugation is given in angular and exact tricyclic coordinates, respectively, by*

$$(\psi_1, \psi_2, \psi_3) \mapsto (-\psi_1 + \theta_1, -\psi_2 + \theta_2, -\psi_3 + \theta_3)$$

and

$$(c_1, c_2, c_3) \mapsto \left(R^2\, c_1^{-1}, R^2\, c_2^{-1}, R^2\, c_3^{-1}\right)$$

Proof. Isogonal conjugation acts on homogeneous trilinear coordinates as $\ell' = \ell^{-1}$ ([1, 273]). By Eq. (2.7), it follows that isogonal conjugation acts on homogeneous tricyclic coordinates as $c' = c^{-1}$. The action on exact tricyclic coordinates can be deduced by observing that if (c_1, c_2, c_3) is exact, then Eq. (2.2) implies $(R^2\, c_1^{-1}, R^2\, c_2^{-1}, R^2\, c_3^{-1})$ is exact.

By Lemma 1.3,

$$\cot(\psi') = \frac{M - c'}{L} = \frac{M - R^2\, c^{-1}}{L} = \cot(\theta - \psi).$$

Hence $\psi' = -\psi + \theta$. □

Remark 3.3. Note that Proposition 3.2 implies that isogonal conjugation preserves Bailey circles. This can be seen directly as follows: Let P and P' be isogonal conjugates and let $\mathcal{C}$ and $\mathcal{C}'$ denote the circles ABP and ABP'. Moving P along $\mathcal{C}$ will rotate $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ some common angle about A and B, respectively. Hence the reflections $\overleftrightarrow{AP'}$ and $\overleftrightarrow{BP'}$ are rotated opposite that angle, which has the effect of moving P' about the circle $\mathcal{C}'$.

3.3. Inversion in the Circumcircle

Inversion maps a point P in the plane to $P \cdot R^2/|P|^2$. It is well known that this transformation preserves the set of circles and lines in the plane, and it clearly fixes A, B, and C. Therefore it must also preserve Bailey circles.

Proposition 3.4. *Inversion in the circumcircle is given in angular and exact tricyclic coordinates, respectively, by*

$$(\psi_1, \psi_2, \psi_3) \mapsto (-\psi_1 + 2\theta_1, -\psi_2 + 2\theta_2, -\psi_3 + 2\theta_3)$$

and

$$(c_1, c_2, c_3) \mapsto \left(\frac{R^2\, c_1}{2M_1 c_1 - R^2}, \frac{R^2\, c_2}{2M_2 c_2 - R^2}, \frac{R^2\, c_3}{2M_3 c_3 - R^2} \right)$$

Proof. A Bailey circle with radius r intersects $E^\perp$ at two points; the coordinates of these points along $E^\perp$ are $c \pm r$. If P is any point on $E^\perp$ with coordinate p, then its inverse P' has coordinate $R^2\, p^{-1}$. Therefore c' is exactly halfway between $R^2(c \pm r)^{-1}$:

$$c' = \frac{1}{2}\left(\frac{R^2}{c-r} + \frac{R^2}{c+r}\right) = \frac{R^2\, c}{c^2 - r^2}.$$

As is evident from Fig. 2,

$$\begin{aligned} r^2 &= (c-M)^2 + L^2 \\ &= c^2 - 2Mc + M^2 + L^2 \\ &= c^2 - 2Mc + R^2. \end{aligned}$$

Therefore $c' = \dfrac{R^2\, c}{2Mc - R^2}$. Equivalently, $R^2\, c^{-1} + R^2\, c'^{-1} = 2M$.

Applying Lemma 1.3 to the equation $R^2\, c^{-1} + R^2\, c'^{-1} = 2M$ yields

$$(M - L\cot(\theta - \psi)) + (M - L\cot(\theta - \psi')) = 2M.$$

It follows that $-\cot(\theta - \psi') = \cot(\theta - \psi)$, so modulo π, $-(\theta - \psi') = \theta - \psi$. Therefore $\psi' = -\psi + 2\theta$. ∎

3.4. Bailey's Theorem and Dihedral Groups

The following theorem is not new, appearing as [2, Theorem 5] and [4, Theorem 13], and it is also proved in [10]. Here we give a proof using angular coordinates.

Theorem 3.5. *Isogonal conjugation maps inverse points to antigonal conjugates.*

Proof. Let a denote antigonal conjugation, s isogonal conjugation, and v inversion. By Proposition 3.1, Proposition 3.2, and Proposition 3.4,

$$(s \circ v)(\psi) = s(-\psi + 2\theta) = -(-\psi + 2\theta) + \theta = \psi - \theta$$

and

$$(a \circ s)(\psi) = a(-\psi + \theta) = -(-\psi + \theta) = \psi - \theta.$$

Therefore $s \circ v = a \circ s$, which is equivalent to the statement of the theorem. □

The next theorem characterizes the group generated by isogonal conjugation and inversion.

Theorem 3.6. *Let s denote isogonal conjugation and v inversion. If the interior angles of ΔABC are rational multiples of π, written in lowest terms as $\theta_i = \pi k_i / n_i$, then*

(i) *The order of $v \circ s$ is $n = \mathrm{lcm}(n_1, n_2, n_3)$,*
(ii) *v and s generate the dihedral group of order $2n$.*

If the interior angles of ΔABC are not all rational multiples of π, then

(i) *$v \circ s$ has infinite order,*
(ii) *v and s generate the infinite dihedral group.*

Proof. By Proposition 3.2 and Proposition 3.4,

$$(v \circ s)(\psi_1, \psi_2, \psi_3) = (\psi_1 + \theta_1, \psi_2 + \theta_2, \psi_3 + \theta_3),$$

so in the first case,

$$(v \circ s)^n(\psi_1, \psi_2, \psi_3) = (\psi_1 + \pi \frac{nk_1}{n_1}, \psi_2 + \pi \frac{nk_2}{n_2}, \psi_3 + \pi \frac{nk_3}{n_3}).$$

The smallest n such that each $\pi n k_i / n_i$ is a multiple of π is $\mathrm{lcm}(n_1, n_2, n_3)$. The second case is clear, since if $n\theta_i$ is a multiple of π for some n, then θ_i is a rational multiple of π. □

Remark 3.7. Inversion may be replaced with antigonal conjugation in the statement of Theorem 3.6 with no significant change in the proof.

4. Resolution of Singularities

The set of angular or exact tricyclic coordinate triples and the set of points in the plane are not in one-to-one correspondence, as outlined in Remark 1.6.

First, points on the circumcircle other than A, B, and C all have the same representation, since the only Bailey circle through such points is the circumcircle. That is, $c_1 = c_2 = c_3 = 0$. This suggests that the circumcircle should be collapsed to a point.

Second, the vertices of ΔABC have ambiguous representation: If $P = A$, for example, then the Bailey circle for BC must be the circumcircle. But, as suggested by Proposition 2.2, any other two Bailey circles which are tangent at A will yield a configuration given by a triple (c_1, c_2, c_3) satisfying Eq. (2.2). The fact that there is one triple of exact coordinates for each line of tangency through A suggests that the plane should be blown up at A: This replaces A

with its exceptional divisor $E_A \simeq \mathbb{RP}^1$, representing each direction through A.

With these issues in mind, we carry out the following steps: First extend the plane to include the line at infinity and blow up each vertex of ΔABC. Then collapse the circumcircle to a point P_0 and the line at infinity to a point P_∞. We will see that the result is a torus, that points in this torus are in one-to-one correspondence with angular and exact tricyclic coordinate triples, and that any birational automorphism of the plane that preserves Bailey circles is a regular automorphism of this torus, with no singularities.

4.1. Construction

We will consider homogeneous tricyclic coordinates as elements of $\mathbb{RP}^1$, written as $[c_i : d_i]$. Denote $[1:0]$ by ∞ and $[0:1]$ by 0. Let T^3_c be the 3-torus of all possible coordinate triples.

The surface of exact tricyclic coordinates given by Eq. (2.2) is defined on that subset $\mathbb{R}^3 \subset T^3_c$ where $d_1 = d_2 = d_3 = 1$. Its closure $\Sigma \subset T^3_c$ is given by the equation

$$R(L_1\, c_1d_2d_3 + L_2\, d_1c_2d_3 + L_3\, d_1d_2c_3) = L_1\, d_1c_2c_3 + L_2\, c_1d_2c_3 + L_3\, c_1c_2d_3.$$

The surface Σ is in fact a 2-torus. This can be seen as follows: First, the set of all triples (ψ_1, ψ_2, ψ_3) is a 3-torus T^3_ψ. The surface of angular coordinates $\Sigma_\psi \subset T^3_\psi$ is given by $\psi_1 + \psi_2 + \psi_3 = 0$, which is a 2-torus. Extending Lemma 1.4 as

$$[c : d] = [R\sin(\psi - \theta) : \sin(\psi)]$$

defines an isomorphism $T^3_c \xrightarrow{\sim} T^3_\psi$ which restricts to $\Sigma \xrightarrow{\sim} \Sigma_\psi$.

The conversion to Cartesian coordinates, Eq. (2.8), extends to a rational map $T^3_c \dashrightarrow \mathbb{RP}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\, d_1\, c_2\, c_3 \\ L_2\, c_1\, d_2\, c_3 \\ L_3\, c_1\, c_2\, d_3 \end{pmatrix}. \tag{4.1}$$

This map is undefined only when $d_1c_2c_3 = c_1d_2c_3 = c_1c_2d_3 = 0$. This occurs only at (∞, ∞, ∞) and the lines $(-, 0, 0)$, $(0, -, 0)$, and $(0, 0, -)$.

Let Φ denote Eq. (4.1) restricted to Σ. Then Φ is undefined only at (∞, ∞, ∞) and $(0, 0, 0)$. Note that Φ does not extend to (∞, ∞, ∞) because, away from this point, $d_1 = 0$, $d_2 = 0$, and $d_3 = 0$ map to the sidelines BC, AC, and AB, respectively, which have no point in common. Similarly, Φ does not extend to $(0, 0, 0)$ because, away from this point, $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$ collapse to A, B, and C, respectively.

Let $\mathring{\Sigma} \subset \Sigma$ be given by removing (∞, ∞, ∞) and the curves $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Let $\mathring{\mathbb{RP}}^2 \subset \mathbb{RP}^2$ be given by removing the line at infinity and the circumcircle. The geometric definition of exact tricyclic coordinates shows that Φ restricts to an isomorphism $\mathring{\Sigma} \to \mathring{\mathbb{RP}}^2$.

Lemma 4.1. *Let $\widetilde{\mathbb{RP}^2}$ denote the blowup of $\mathbb{RP}^2$ at the vertices A, B, and C. Then Φ lifts to a rational map $\widetilde{\Phi}$ as in the following diagram:*

$$\begin{array}{ccc} & & \widetilde{\mathbb{RP}^2} \\ & \overset{\widetilde{\Phi}}{\nearrow} & \downarrow \\ \Sigma & \underset{\Phi}{\dashrightarrow} & \mathbb{RP}^2 \end{array}$$

The map $\widetilde{\Phi}$ is undefined only at (∞, ∞, ∞) and $(0, 0, 0)$. Let $\mathring{Z}_1$ denote the curve $c_1 = 0$ minus the point $(0, 0, 0)$, and define $\mathring{Z}_2$ and $\mathring{Z}_3$ analogously. Let $\mathring{E}_A$ denote the exceptional divisor of A minus the direction tangent to the circumcircle, and define $\mathring{E}_B$ and $\mathring{E}_C$ analogously. Then whereas Φ collapses each $\mathring{Z}_i$ to a point, $\widetilde{\Phi}$ restricts to isomorphisms

$$\mathring{Z}_1 \xrightarrow{\sim} \mathring{E}_A, \quad \mathring{Z}_2 \xrightarrow{\sim} \mathring{E}_B, \quad \mathring{Z}_3 \xrightarrow{\sim} \mathring{E}_C.$$

Proof. There is clearly a rational map $\widetilde{\Phi} : \Sigma \dashrightarrow \widetilde{\mathbb{RP}^2}$ that is identical to Φ, excluding from the domain those points mapping to A, B, or C. From Eq. (4.1), it can be deduced that $\Phi^{-1}(A) = \mathring{Z}_1$, and similarly for B and C. What must be shown, then, is that $\widetilde{\Phi}$ extends to $\mathring{Z}_1$, mapping it isomorphically onto $\mathring{E}_A$, and similarly for $\mathring{Z}_2$ and $\mathring{Z}_3$. We will demonstrate the first assertion, the others being analogous.

To understand how $\widetilde{\Phi}$ behaves near $c_1 = 0$, consider the original plane as the open subset of $\widetilde{\mathbb{RP}^2}$ with $z = 1$, and blow up the plane at $A = (x_1, y_1)$. Points in this blowup can be understood as consisting of a point $P = (x, y)$ along with a direction through A, with the restriction that if $P \neq A$, the direction must coincide with $\overleftrightarrow{AP}$. By Eq. (4.1), the direction of $\overleftrightarrow{AP}$ is given by

$$[x - x_1 : y - y_1]^T = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} L_2\, d_2 c_3 \\ L_3\, c_2 d_3 \end{pmatrix}. \tag{4.2}$$

This clearly extends to $\mathring{Z}_1$, and we may consider Eq. (4.2) as a rational map $\mathring{Z}_1 \dashrightarrow \mathring{E}_A$. Now consider the parameterization of $c_1 = 0$ in Σ by the parameter $[u : v] \in \mathbb{RP}^1$:

$$\begin{aligned} [c_1 : d_1] &= [0 : 1] \\ [c_2 : d_2] &= [L_3^2\, u + L_2^2\, v : R^{-1} L_1 L_3\, u] \\ [c_3 : d_3] &= [L_3^2\, u + L_2^2\, v : R^{-1} L_1 L_2\, v] \end{aligned}$$

Away from $[u : v] = [L_2^2 : -L_3^2]$, which corresponds to $(0, 0, 0)$, it is easily verified that

$$[L_2\, d_2 c_3 : L_3\, c_2 d_3] = [u : v].$$

It follows that $\widetilde{\Phi}$ maps the point $[u : v]$ on $c_1 = 0$ to the direction $u\overrightarrow{AB} + v\overrightarrow{AC}$ through A. The missing direction $L_2^2\overrightarrow{AB} - L_3^2\overrightarrow{AC}$ is tangent to the circumcircle, as shown below. So Eq. (4.2) is in fact an isomorphism $\mathring{Z}_1 \xrightarrow{\sim} \mathring{E}_A$.

Let P be a point in the plane and suppose that $\overleftrightarrow{AP}$ is tangent to the circumcircle. Write

$$\overrightarrow{AP} = u\overrightarrow{AB} + v\overrightarrow{AC}.$$

Then

$$0 = \overrightarrow{OA} \cdot \overrightarrow{AP} = u\,\overrightarrow{OA} \cdot \overrightarrow{AB} + v\,\overrightarrow{OA} \cdot \overrightarrow{AC},$$

so

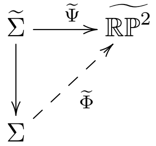
$$[u : v] = [-\overrightarrow{OA} \cdot \overrightarrow{AC} : \overrightarrow{OA} \cdot \overrightarrow{AB}].$$

Now observe that

$$\begin{aligned} 4L_2^2 = |\overrightarrow{AC}|^2 &= (\overrightarrow{OC} - \overrightarrow{OA}) \cdot (\overrightarrow{OC} - \overrightarrow{OA}) \\ &= 2R^2 - 2\,\overrightarrow{OA} \cdot \overrightarrow{OC} \\ &= 2\,\overrightarrow{OA} \cdot (\overrightarrow{OA} - \overrightarrow{OC}) = -2\,\overrightarrow{OA} \cdot \overrightarrow{AC}. \end{aligned}$$

Similarly, $4L_3^2 = |\overrightarrow{AB}|^2 = -2\,\overrightarrow{OA} \cdot \overrightarrow{AB}$. Therefore $[u : v] = [L_2^2 : -L_3^2]$. □

Lemma 4.2. *Let $\widetilde{\Sigma}$ denote the blowup of Σ at (∞, ∞, ∞) and $(0, 0, 0)$. Then $\widetilde{\Phi}$ extends to an isomorphism $\widetilde{\Psi}$ as in the following diagram:*



The exceptional divisors of (∞, ∞, ∞) and $(0, 0, 0)$ map isomorphically via $\widetilde{\Psi}$ onto the line at infinity and the proper transform of the circumcircle in $\widetilde{\mathbb{RP}^2}$, respectively.

Proof. Let E_∞ and E_0 denote the exceptional divisors of (∞, ∞, ∞) and $(0, 0, 0)$, respectively, in $\widetilde{\Sigma}$. Observe that there is a rational map $\widetilde{\Psi} : \widetilde{\Sigma} \dashrightarrow \widetilde{\mathbb{RP}^2}$ which is undefined on E_∞ and E_0 but is otherwise identical to $\widetilde{\Phi}$. What must be shown, then, is that this map $\widetilde{\Psi}$ extends to E_∞ and E_0 as in the statement of the lemma.

First we determine how $\widetilde{\Psi}$ behaves near E_∞. Consider the open subset $V \subset T_c^3$ obtained by dehomogenizing at $c_i = 1$ for each i. Then $V \simeq \mathbb{R}^3_{(d_1, d_2, d_3)}$ and the point $(\infty, \infty, \infty) \in T_c^3$ is given by $\hat{0} \in V$. Let $\widetilde{V}$ denote the blowup of V at $\hat{0}$. That is, $\widetilde{V}$ is the set of

$$\big((d_1, d_2, d_3),\ [\beta_1 : \beta_2 : \beta_3]\big) \in V \times \mathbb{RP}^2$$

satisfying $[\beta_1 : \beta_2 : \beta_3] = [d_1 : d_2 : d_3]$ when $\hat{d} \neq \hat{0}$. The map $\widetilde{V} \dashrightarrow \mathbb{RP}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\,\beta_1 \\ L_2\,\beta_2 \\ L_3\,\beta_3 \end{pmatrix} \tag{4.3}$$

agrees with Eq. (4.1) away from $\hat{d} = \hat{0}$. Now consider the intersection of Σ with V, defined by the equation

$$L_1d_1 + L_2d_2 + L_3d_3 - R(L_1d_2d_3 + L_2d_1d_3 + L_3d_1d_2) = 0.$$

The tangent plane to Σ at $\hat{0} \in V$ is given by $L_1d_1 + L_2d_2 + L_3d_3 = 0$, so E_∞ sits inside of $\{\hat{0}\} \times \mathbb{RP}^2 \subset \widetilde{V}$ as $L_1\beta_1 + L_2\beta_2 + L_3\beta_3 = 0$. Clearly Eq. (4.3) extends to E_∞, and in fact maps it isomorphically onto $z = 0$, the line at infinity. Moreover, this determines the extension of $\widetilde{\Psi}$ to E_∞, since the image of E_∞ under Eq. (4.3) does not include A, B, or C.

We now determine the behavior of $\widetilde{\Psi}$ near E_0. Consider the open subset $U \subset T_c^3$ obtained by dehomogenizing at $d_i = 1$ for each i. Then $U \simeq \mathbb{R}^3_{(c_1,c_2,c_3)}$ and the point $(0,0,0) \in T_c^3$ is given by $\hat{0} \in U$. Let $\widetilde{U}$ denote the blowup of U at $\hat{0}$, which is the set of

$$\big((c_1, c_2, c_3),\ [\alpha_1 : \alpha_2 : \alpha_3]\big) \in U \times \mathbb{RP}^2$$

satisfying $[\alpha_1 : \alpha_2 : \alpha_3] = [c_1 : c_2 : c_3]$ when $\hat{c} \neq \hat{0}$. The map $\widetilde{U} \dashrightarrow \mathbb{RP}^2$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} L_1\,\alpha_2\alpha_3 \\ L_2\,\alpha_1\alpha_3 \\ L_3\,\alpha_1\alpha_2 \end{pmatrix} \tag{4.4}$$

agrees with Eq. (4.1) away from $\hat{c} = \hat{0}$. The intersection of Σ with U is defined by the equation

$$L_1c_1 + L_2c_2 + L_3c_3 - R^{-1}(L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2) = 0.$$

The tangent plane to Σ at $\hat{0} \in U$ is given by $L_1c_1 + L_2c_2 + L_3c_3 = 0$, so E_0 sits inside of $\{\hat{0}\} \times \mathbb{RP}^2 \subset \widetilde{U}$ as $L_1\alpha_1 + L_2\alpha_2 + L_3\alpha_3 = 0$. Observe that Eq. (4.4) extends to E_0.

The image of E_0 under Eq. (4.4) includes A, B, and C, so to determine $\widetilde{\Psi}$, we must also determine a direction through A, B, and C. Again assuming $\hat{c} \neq \hat{0}$, Eq. (4.2) shows that the direction through A is given by

$$L_2\alpha_3\overrightarrow{AB} + L_3\alpha_2\overrightarrow{AC}. \tag{4.5}$$

This extends to E_0 as well, and directions through the other vertices can be determined similarly. We have therefore determined how $\widetilde{\Psi}$ extends to E_0, but it remains to be shown that it maps E_0 isomorphically onto the proper transform of the circumcircle.

By Eq. (2.6) and Eq. (2.5), assuming exact tricyclic coordinates,

$$R^2 - |OP|^2 = \frac{2\,|\Delta ABC|\,c_1c_2c_3}{L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2}.$$

For homogeneous tricyclic coordinates, each c_i must be scaled according to Eq. (2.4). This yields

$$R^2 - |OP|^2 = \frac{2R\,|\Delta ABC|\,c_1c_2c_3(L_1c_1 + L_2c_2 + L_3c_3)}{(L_1c_2c_3 + L_2c_1c_3 + L_3c_1c_2)^2}.$$

It follows that away from $\hat{0}$ in U, the surface $L_1c_1 + L_2c_2 + L_3c_3 = 0$ maps via Eq. (4.1) to the circumcircle. By continuity, $\widetilde{\Psi}$ must therefore map E_0 to the proper transform of the circumcircle. It remains to be seen that it is an isomorphism.

Observe that $[0 : -L_3 : L_2]$ is the only point of E_0 with $\alpha_1 = 0$, and therefore the only point of E_0 mapped by Eq. (4.4) to A. By Eq. (4.5), it is sent to the direction $L_2^2\overrightarrow{AB} - L_3^2\overrightarrow{AC}$ in the exceptional divisor of A which, as discussed in Lemma 4.1, is the direction tangent to the circumcircle. The same principle holds for the two points of E_0 with $\alpha_2 = 0$ and $\alpha_3 = 0$, respectively.

Finally, let $\mathring{E}_0$ denote E_0 minus the three points with $\alpha_1 = 0$, $\alpha_2 = 0$, and $\alpha_3 = 0$, and let $\mathring{\mathcal{C}}$ denote the proper transform of the circumcircle minus the three points over the vertices A, B, and C. All that remains to be shown is that the map $\mathring{E}_0 \to \mathring{\mathcal{C}}$ induced by Eq. (4.4) is an isomorphism. This is true, since Eq. (4.4) is invertible away from $\alpha_1\alpha_2\alpha_3 = 0$. □

Theorem 4.3. *Suppose the line at infinity and the proper transform of the circumcircle in $\widetilde{\mathbb{RP}^2}$ are collapsed to points P_∞ and P_0, respectively, yielding a surface T. Then the isomorphism $\widetilde{\Psi}$ descends to an isomorphism Ψ as in the following diagram:*

$$\begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{\widetilde{\Psi}} & \widetilde{\mathbb{RP}^2} \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\Psi} & T \end{array}$$

The points (∞, ∞, ∞) and $(0, 0, 0)$ map via Ψ to P_∞ and P_0, respectively.

Proof. This follows immediately from Lemma 4.2. □

Remark 4.4. A slight modification of Theorem 4.3 illustrates Dyck's theorem: If E_∞ is collapsed to a point in $\widetilde{\Sigma}$, the result is the same as Σ blown up at one point; topologically, this is $T^2\#\mathbb{RP}^2$. If the line at infinity is collapsed to a point in $\widetilde{\mathbb{RP}^2}$, the result is the same as a sphere blown up at three points; topologically this is $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$. The isomorphism $\widetilde{\Psi}$ descends to an isomorphism between these surfaces.

4.2. Features on the Torus

The construction of the surface T is described topologically in Fig. 5 and Fig. 6. The curves $d_i = 0$ in Σ map onto the sidelines of ΔABC. These are represented by dashed lines in Fig. 5 and Fig. 6. The curves $c_i = 0$ in Σ map onto the exceptional divisors of A, B, and C. These are represented by solid lines in Fig. 5 and Fig. 6.

Remark 4.5. Points in the surface Σ are in one-to-one correspondence with configurations of Bailey circles (ψ_1, ψ_2, ψ_3) satisfying $\psi_1+\psi_2+\psi_3 = 0$. Recall that Proposition 2.2 classified such configurations: Case (i), in which all three Bailey circles are sidelines, corresponds to P_∞. Case (ii), in which one Bailey

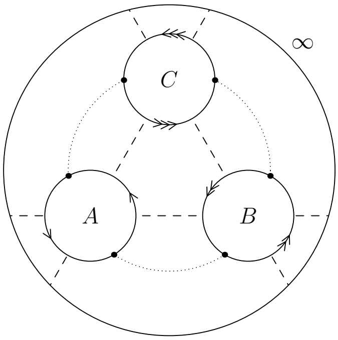


FIGURE 5. Blowing up at A, B, and C.

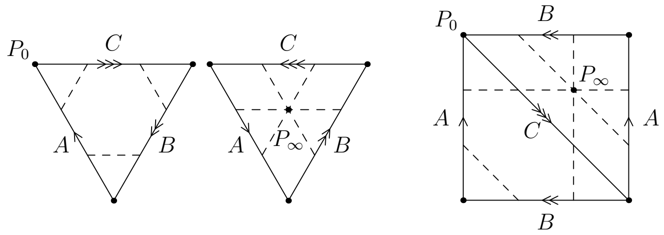


FIGURE 6. Inside and outside the circumcircle; the torus T.

circle is the circumcircle and the other two are tangent, corresponds to the solid lines in Fig. 5 and Fig. 6; the special case that all three Bailey circles are the circumcircle corresponds to the intersection P_0.

In Proposition 4.6 we summarize the coordinates for several points on T. The angular coordinates in this list agree with those found in [9]. Hofstadter points and related centers are investigated in Section 5.4. Other points, including Brocard points, isodynamic points, and isogonic centers, fit naturally into the perspective of Bailey circles and angular coordinates, but we omit these from the list.

Proposition 4.6. *Let H, O, and I denote the orthocenter, circumcenter, and incenter, respectively. The following table shows angular and exact tricyclic coordinates.*

	c	ψ
P_∞	∞	0
P_0	0	θ
H	$2M$	$-\theta$
O	$R^2/(2M)$	2θ
I	R	$(\theta+\pi)/2$

Proof. The coordinates for P_∞ and P_0 follow directly from Theorem 4.3. The angular coordinates for H, O, and I can be determined as follows:

In Theorem 2.11 with $\psi = -\theta$, the triangle $\Delta A'BC$ is exactly ΔABC reflected over the sideline $\overleftrightarrow{BC}$. So $\overleftrightarrow{AA'}$ must be perpendicular to $\overleftrightarrow{BC}$. Similarly $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are perpendicular to $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$, respectively, so $P = H$.

The angular coordinate ψ_1 for O is $\measuredangle BOC$; by the inscribed angle theorem this is equal to $2\measuredangle BAC = 2\theta_1$. The angular coordinate ψ_1 for I is $\measuredangle BIC = \pi - \theta_2/2 - \theta_3/2$. This is equal (modulo π, as usual) to $(\theta_1 + \pi)/2$. The coordinates ψ_2 and ψ_3 can be deduced similarly.

The exact tricyclic coordinates can be determined using Lemma 1.4. □

Birational automorphisms of the plane which preserve Bailey circles can be viewed as automorphisms of the torus T. One the one hand, when viewed as acting on the plane,

(i) Antigonal conjugation is not defined at the orthocenter H,
(ii) Isogonal conjugation is not defined on the circumcircle,
(iii) Inversion in the circumcircle is not defined at the circumcenter O.

One the other hand, when viewed as automorphisms of T, Proposition 3.1, Proposition 3.2, Proposition 3.4, and Proposition 4.6 show that

(i) Antigonal conjugation exchanges $H \leftrightarrow P_0$ and fixes P_∞,
(ii) Isogonal conjugation exchanges $P_0 \leftrightarrow P_\infty$ and fixes I,
(iii) Inversion in the circumcircle exchanges $O \leftrightarrow P_\infty$ and fixes P_0.

5. Translations, Reflections, and Triangle Centers

5.1. Classification

Consider a point P with angular coordinates $(\alpha_1, \alpha_2, \alpha_3)$. Then the map

$$\psi \mapsto -(\psi - \alpha) + \alpha \tag{5.1}$$

fixes P. We will refer to this map as *angular reflection about* P. By Proposition 3.1, Proposition 3.2, and Proposition 3.4, we see that the following maps are angular reflections:

(i) Antigonal conjugation: Reflection about P_∞ $(\alpha = 0)$
(ii) Isogonal conjugation: Reflection about the incenter $(\alpha = (\theta + \pi)/2)$
(iii) Inversion in the circumcircle: Reflection about P_0 $(\alpha = \theta)$

Let ω_1, ω_2, and ω_3 be any triple satisfying $\omega_1 + \omega_2 + \omega_3 = 0$. The map

$$\psi \mapsto \psi + \omega \tag{5.2}$$

will be referred to as an *angular translation*. The maps given by isogonal conjugation followed by inversion or isogonal conjugation followed by antigonal conjugation, as in Theorem 3.5 and Theorem 3.6, are angular translations.

Angular translations and angular reflections are given in terms of exact tricyclic coordinates in Proposition 5.1.

Proposition 5.1. *The angular translation Eq.* (5.2) *acts on the exact tricyclic coordinate c by the Möbius transformation*

$$\begin{pmatrix} R\sin(\theta-\omega) & R^2\sin(\omega) \\ -\sin(\omega) & R\sin(\theta+\omega) \end{pmatrix} \tag{5.3}$$

and the angular reflection Eq. (5.1) *acts on the exact tricyclic coordinate c by the Möbius transformation*

$$\begin{pmatrix} R\sin(2\alpha-\theta) & -R^2\sin(2\alpha-2\theta) \\ \sin(2\alpha) & -R\sin(2\alpha-\theta) \end{pmatrix}. \tag{5.4}$$

Proof. Let $z = e^{2i\psi}$ and $\Theta = e^{i\theta}$. By Eq. (2.3), z is equal to the Möbius transformation $\begin{pmatrix} -1 & R\Theta \\ -1 & R\overline{\Theta} \end{pmatrix}$ applied to c. Hence c is equal to the Möbius transformation $\begin{pmatrix} R\overline{\Theta} & -R\Theta \\ 1 & -1 \end{pmatrix}$ applied to z.

Now let $\Omega = e^{i\omega}$. The action of Eq. (5.2) on z is given by

$$z \mapsto e^{2i(\psi+\omega)} = z\Omega^2 = z\Omega/\overline{\Omega},$$

which is the Möbius transformation $\begin{pmatrix} \Omega & 0 \\ 0 & \overline{\Omega} \end{pmatrix}$. The action on c is therefore

$$\begin{aligned} &\frac{1}{2i}\begin{pmatrix} R\overline{\Theta} & -R\Theta \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \Omega & 0 \\ 0 & \overline{\Omega} \end{pmatrix}\begin{pmatrix} -1 & R\Theta \\ -1 & R\overline{\Theta} \end{pmatrix} \\ &= \frac{1}{2i}\begin{pmatrix} R(\Theta\overline{\Omega}-\overline{\Theta}\Omega) & R^2(\Omega-\overline{\Omega}) \\ \overline{\Omega}-\Omega & R(\Theta\Omega-\overline{\Theta\Omega}) \end{pmatrix}. \end{aligned}$$

The entries of the matrix are the imaginary parts of $R^2\Omega$, $\overline{\Omega}$, $R\Theta\Omega$, and $R\Theta\overline{\Omega}$.

Finally, let $A = e^{2i\alpha}$. The action of Eq. (5.1) on z is given by

$$z \mapsto e^{2i(-\psi+2\alpha)} = \overline{z}A^2 = A/(z\overline{A}),$$

which is the Möbius transformation $\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}$. Following the same reasoning as above, the action on c is given by

$$\frac{1}{2i}\begin{pmatrix} R(A\overline{\Theta}-\overline{A}\Theta) & -R^2(A\overline{\Theta}^2-\overline{A}\Theta^2) \\ A-\overline{A} & -R(A\overline{\Theta}-\overline{A}\Theta) \end{pmatrix}.$$

The entries of the matrix are the imaginary parts of A, $\pm RA\overline{\Theta}$, and $-R^2A\overline{\Theta}^2$. □

Every angular reflection and angular translation is an automorphism of the torus of angular (or exact tricyclic) coordinates, and can be viewed as a birational automorphism of the plane which preserves Bailey circles. In Theorem 5.2 we show that, in fact, every birational automorphism of the plane which preserves Bailey circles must be an angular reflection or an angular translation.

Theorem 5.2. *Any birational automorphism of the plane which preserves Bailey circles must be an angular reflection or an angular translation.*

Proof. Let F be any birational automorphism of the plane which preserves Bailey circles. Recall that such map may be written in the form Eq. (3.1). Moreover, by Eq. (2.8), each f_i must be a birational map of one variable, hence a Möbius transformation.

Let $z = e^{2i\psi}$ and $\Theta = e^{i\theta}$. By the same reasoning used in Proposition 5.1, f_i must be given by the Möbius transformation in z with matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{R^2}\begin{pmatrix} -1 & R\Theta \\ -1 & R\overline{\Theta} \end{pmatrix}\begin{pmatrix} A\,R & B\,R^2 \\ C & D\,R \end{pmatrix}\begin{pmatrix} R\overline{\Theta} & -R\Theta \\ 1 & -1 \end{pmatrix} \tag{5.5}$$
$$= \begin{pmatrix} -(A\overline{\Theta}+B)+\Theta(C\overline{\Theta}+D) & (A\Theta+B)-\Theta(C\Theta+D) \\ -(A\overline{\Theta}+B)+\overline{\Theta}(C\overline{\Theta}+D) & (A\Theta+B)-\overline{\Theta}(C\Theta+D) \end{pmatrix},$$

where the middle matrix of Eq. (5.5) defines f_i as a Möbius transformation in c_i. Observe that $\beta = -\overline{\alpha} + \overline{\zeta}$, $\gamma = \alpha - \zeta$, and $\delta = -\overline{\alpha}$, where $\zeta = (\Theta - \overline{\Theta})(C\overline{\Theta} + D)$. Moreover, $\zeta \neq 0$ since θ is not a multiple of π.

The transformation F must map exact triples to exact triples. Exactness is equivalent to $z_1z_2z_3 = e^{2i(\psi_1+\psi_2+\psi_3)} = 1$. Similarly, $z_1'z_2'z_3' = 1$, so

$$1 = z_1'z_2'z_3' = \frac{(\alpha_1z_1+\beta_1)(\alpha_2z_2+\beta_2)(\alpha_3z_3+\beta_3)}{(\gamma_1z_1+\delta_1)(\gamma_2z_2+\delta_2)(\gamma_3z_3+\delta_3)}.$$

Therefore

$$(\alpha_1z_1+\beta_1)(\alpha_2z_2+\beta_2)(\alpha_3z_3+\beta_3)-(\gamma_1z_1+\delta_1)(\gamma_2z_2+\delta_2)(\gamma_3z_3+\delta_3)=0.$$

Expanding, replacing $z_3 = 1/(z_1z_2)$, and multiplying through by z_1z_2 yields:

$$\begin{aligned}
\big[(\alpha_1\alpha_2\alpha_3-\gamma_1\gamma_2\gamma_3) \;+\; (\beta_1\beta_2\beta_3-\delta_1\delta_2\delta_3)\big]\,z_1z_2 \;+\\
[\alpha_1\alpha_2\beta_3-\gamma_1\gamma_2\delta_3]\,z_1^2z_2^2 \;+\; [\alpha_1\beta_2\alpha_3-\gamma_1\delta_2\gamma_3]\,z_1 \;+\\
[\alpha_1\beta_2\beta_3-\gamma_1\delta_2\delta_3]\,z_1^2z_2 \;+\; [\beta_1\alpha_2\alpha_3-\delta_1\gamma_2\gamma_3]\,z_2 \;+\\
[\beta_1\alpha_2\beta_3-\delta_1\gamma_2\delta_3]\,z_1z_2^2 \;+\; [\beta_1\beta_2\alpha_3-\delta_1\delta_2\gamma_3] \;=\; 0.
\end{aligned}$$

That is, the polynomial on the left-hand side must vanish whenever $|z_1| = |z_2| = 1$. But this implies that the polynomial is identically zero, so each quantity in square brackets must be zero. In particular,

$$\begin{aligned}
(\overline{\alpha_1}-\overline{\zeta_1})\alpha_2\alpha_3 &= \overline{\alpha_1}(\alpha_2-\zeta_2)(\alpha_3-\zeta_3)\\
\alpha_1(\overline{\alpha_2}-\overline{\zeta_2})\alpha_3 &= (\alpha_1-\zeta_1)\overline{\alpha_2}(\alpha_3-\zeta_3)\\
\alpha_1\alpha_2(\overline{\alpha_3}-\overline{\zeta_3}) &= (\alpha_1-\zeta_1)(\alpha_2-\zeta_2)\overline{\alpha_3}
\end{aligned}$$

Observe that if $\alpha_i = 0$ for any i, then $\alpha_i = 0$ for all i. Similarly, if $\alpha_i = \zeta_i$ for any i, then $\alpha_i = \zeta_i$ for all i. In the first case, the matrix Eq. (5.5) is $\begin{pmatrix} 0 & \overline{\zeta} \\ -\zeta & 0 \end{pmatrix}$. In the second case, the matrix Eq. (5.5) is $\begin{pmatrix} \zeta & 0 \\ 0 & -\overline{\zeta} \end{pmatrix}$. These are angular reflection and angular translation, respectively.

Finally, suppose $\alpha_i \neq 0, \zeta_i$ for any i. Let $\xi = (\alpha - \zeta)/\alpha$. Then the equations above can be rewritten as:

$$\overline{\xi_1} = \xi_2\xi_3, \;\; \overline{\xi_2} = \xi_1\xi_3, \;\; \overline{\xi_3} = \xi_1\xi_2.$$

Therefore $\xi_3 = \overline{\xi_1\xi_2} = \overline{\xi_1}\xi_1\xi_3$. Since $\xi_3 \neq 0$, it follows that $|\xi_1| = 1$. Repeating this argument yields

$$|\xi_1| = |\xi_2| = |\xi_3| = 1.$$

Hence $|\alpha| = |\alpha - \zeta|$, so there exists ρ such that $|\rho| = 1$ and $\alpha - \zeta = \rho\alpha$. The matrix Eq. (5.5) is therefore $\begin{pmatrix} \alpha & -\overline{\alpha\rho} \\ \alpha\rho & -\overline{\alpha} \end{pmatrix}$. But this is the constant map $z \mapsto \overline{\rho}$, which is impossible. □

5.2. Angular Reflections

Inversion in the circumcircle exchanges the regions inside and outside the circumcircle. In [2, Theorem 6], it is proved that antigonal conjugation also exchanges two regions, with the sidelines acting as boundaries between these. These are special cases of a more general phenomenon: Inversion and antigonal conjugation are both angular reflections; we will show that every angular reflection exchanges two regions of the plane.

The torus of angular coordinates is shown in Fig. 7. The vertical lines have fixed ψ_1 value, the horizontal lines have fixed ψ_2 value, and the diagonal lines have fixed ψ_3 value (since $\psi_1+\psi_2+\psi_3 = 0$). In terms of the isomorphism given in Theorem 4.3, the features of the diagram map to the modified plane in the following way: The dashed lines through P_∞ correspond to the sidelines of ΔABC; the dotted lines through P_0 correspond to the exceptional divisors of A, B, and C (hence these lines collapse to the vertices in the original plane); and the solid lines correspond to the Bailey circles through P.

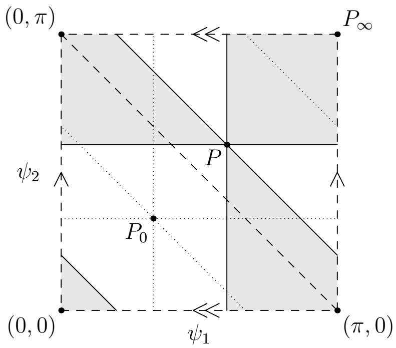


FIGURE 7. Two regions on the torus of angular coordinates.

When viewed as acting on the $\psi_1\psi_2$-plane, angular reflection about P simply reflects each point through P or, equivalently, rotates about P through an angle of π. This exchanges the shaded and unshaded regions shown in Fig. 7. These two regions correspond to regions in the plane; when viewed as acting on the plane, angular reflection about P exchanges these.

Let $\mathcal{R}_T$ denote the shaded region of the torus shown in Fig. 7, and let $\mathcal{R}$ denote the corresponding region in the plane. To understand what $\mathcal{R}$ looks like, observe that containment in $\mathcal{R}_T$ flips exactly when a solid line is crossed. Hence, containment in $\mathcal{R}$ flips exactly when a Bailey circle for P is crossed.

The dotted lines correspond to the exceptional divisors of the vertices, so each point on a dotted line can be interpreted as an infinitesimal line segment through a vertex in the plane. The points of intersection with solid lines correspond to those infinitesimal line segments tangent to a Bailey circle for P. Following along a dotted line in the torus, containment in $\mathcal{R}_T$ flips exactly when a solid line is crossed. Hence, as an infinitesimal line segment through a vertex is rotated, containment in $\mathcal{R}$ flips exactly when it passes through a direction tangent to a Bailey circle for P.

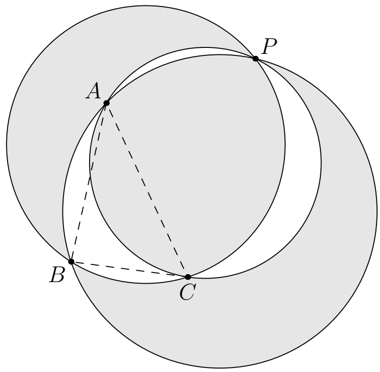


FIGURE 8. Angular reflection about P exchanges the shaded and unshaded regions.

This observation is summarized in Theorem 5.3, leaving reflection about P_0 and about P_∞ as special cases. These cases correspond to inversion in the circumcircle and antigonal conjugation, respectively. The fact that $\mathcal{R}$ is the region inside the circumcircle in the case $P = P_0$ is illustrated by Fig. 5 and Fig. 6. A general example is shown in Fig. 8.

Theorem 5.3. *Let P be a point not on the circumcircle.*

Let $\mathcal{B}$ denote the union of the three Bailey circles for P. Each Bailey circle divides the plane into two open regions; let $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ denote one of those regions for each Bailey circle. Finally, let $\mathcal{R}$ (respectively, $\mathcal{R}'$) denote the regions consisting of those points not in $\mathcal{B}$ which are contained in an even (respectively, odd) number of $\mathcal{D}_i$.

Then angular reflection about P maps $\mathcal{R}$ to $\mathcal{R}'$ and vice-versa.

5.3. Preservation of Triangle Centers

Denote the side lengths of ΔABC by $a = |BC|$, $b = |AC|$, and $c = |AB|$. The following definitions can be found in [7] and [5]. A *triangle center* is a point P with homogeneous trilinear coordinates

$$\big(f(a,b,c) : f(b,c,a) : f(c,a,b)\big)$$

where f is a homogeneous function which is symmetric in the last two variables. The function f is called the *center-function*. Note that different center-functions may produce the same triangle center for every triangle since a center-function produces homogeneous coordinates.

A *polynomial center* can be described by a center-function f which is a rational function in the side lengths alone. A *regular center* has the weaker property that f may be taken to be a rational function in the side lengths and $|\Delta ABC|$. Since $4R|\Delta ABC| = abc$, this is equivalent to f being a rational function in the side lengths and circumradius R.

The following describes a sufficient condition for a plane transformation which depeneds on a triangle ΔABC (such as an angular reflection or angular translation) to map any polynomial triangle center to another polynomial triangle center.

Lemma 5.4. *Let F be a rational function in the variables a, b, c, ℓ_1, ℓ_2, and ℓ_3. For each triangle ΔABC with side lengths a, b, and c, there is an associated plane transformation $(\ell_1 : \ell_2 : \ell_3) \mapsto (\ell_1' : \ell_2' : \ell_3')$, where*

$$\begin{aligned} \ell_1' &= F(a, b, c,\ \ell_1, \ell_2, \ell_3), \\ \ell_2' &= F(b, c, a,\ \ell_2, \ell_3, \ell_1), \\ \ell_3' &= F(c, a, b,\ \ell_3, \ell_1, \ell_2). \end{aligned} \tag{5.6}$$

Suppose F satisfies the following properties:

(i) $F(\lambda a, \lambda b, \lambda c,\ \ell_1, \ell_2, \ell_3) = \lambda^m F(a, b, c,\ \ell_1, \ell_2, \ell_3)$ *for some* $m \in \mathbb{Z}$,
(ii) $F(a, b, c,\ \lambda\ell_1, \lambda\ell_2, \lambda\ell_3) = \lambda^n F(a, b, c,\ \ell_1, \ell_2, \ell_3)$ *for some* $n \in \mathbb{Z}$,
(iii) $F(a, c, b,\ \ell_1, \ell_3, \ell_2) = F(a, b, c,\ \ell_1, \ell_2, \ell_3)$.

Then if P is a polynomial triangle center for ΔABC, the plane transformation Eq. (5.6) *maps P to another polynomial triangle center for ΔABC.*

Proof. If P is a polynomial triangle center with center-function $f(a, b, c)$, then it has trilinear coordinates

$$(\ell_1 : \ell_2 : \ell_3) = \big(f(a, b, c) : f(b, c, a) : f(c, a, b)\big).$$

Substituting these values into the expressions for ℓ_i' in Eq. (5.6) yields

$$(\ell_1' : \ell_2' : \ell_3') = \big(g(a, b, c) : g(b, c, a) : g(c, a, b)\big),$$

where

$$g(a, b, c) = F(a, b, c, f(a, b, c), f(b, c, a), f(c, a, b)).$$

Clearly g is a rational function in a, b, and c only, so all that remains to be shown is that g is homogeneous and is symmetric in the last two variables. This follows from the corresponding properties of f along with conditions (i), (ii), and (iii) satisfied by F. □

Next we will use Lemma 5.4 to show that certain angular translations and angular reflections preserve polynomial triangle centers, but before proceeding we must determine how angular translations and angular reflections act on trilinear coordinates.

Lemma 5.5. *The angular translation Eq.* (5.2) *can be written in homogeneous trilinear coordinates as*

$$\ell \mapsto \ell' = \frac{-\sin(\omega)\Lambda_2 + \sin(\theta+\omega)\Lambda_1\ell}{\sin(\theta-\omega)\Lambda_2 + \sin(\omega)\Lambda_1\ell} \tag{5.7}$$

and the angular reflection Eq. (5.1) *can be written in homogeneous trilinear coordinates as*

$$\ell \mapsto \ell' = \frac{\sin(2\alpha)\Lambda_2 - \sin(2\alpha-\theta)\Lambda_1\ell}{\sin(2\alpha-\theta)\Lambda_2 - \sin(2\alpha-2\theta)\Lambda_1\ell}, \tag{5.8}$$

where $\Lambda_1 = L_1\ell_1 + L_2\ell_2 + L_3\ell_3$ *and* $\Lambda_2 = L_1\ell_2\ell_3 + L_2\ell_1\ell_3 + L_3\ell_1\ell_2$.

Proof. This follows directly from Proposition 5.1 and Proposition 2.10. Note that one factor of R has been dropped from each expression defining ℓ', which is permissible since the coordinates are homogeneous. □

The fact that the circumradius R does not appear in Eq. (5.7) or Eq. (5.8) will be important when considering whether these maps send polynomial centers to polynomial, as opposed to regular, centers.

Lemma 5.6. *Let* h *denote the angular translation Eq.* (5.2) *with* $\omega = (\theta+\pi)/2$. *Then* h *and* h^{-1} *preserve polynomial triangle centers.*

Proof. The angular translation h can be written in homogeneous trilinear coordinates as in Eq. (5.7). Observe that

$$\begin{aligned}\frac{\sin(\theta\pm\omega)}{\sin(\omega)} &= \sin(\theta)\cot(\omega) \pm \cos(\theta)\\ &= -\sin(\theta)\tan(\theta/2) \pm \cos(\theta)\\ &= -(1-\cos(\theta)) \pm \cos(\theta),\end{aligned}$$

so Eq. (5.7) may be written as

$$\ell \mapsto \ell' = \frac{-\Lambda_2 + (2\cos(\theta)-1)\Lambda_1\ell}{-\Lambda_2 + \Lambda_1\ell}.$$

By the law of cosines,

$$\cos(\theta_1) = \frac{L_2^2 + L_3^2 - L_1^2}{2L_2L_3},$$

and similarly for $\cos(\theta_2)$ and $\cos(\theta_3)$. Therefore ℓ'_i is a rational function of L_1, L_2, L_3, ℓ_1, ℓ_2, and ℓ_3. Let $F_i(a,b,c,\ell_1,\ell_2,\ell_3)$ denote this rational function (recalling that $a = 2L_1$, $b = 2L_2$, $c = 2L_3$).

The function F_1 satisfies properties (i) and (ii) of Lemma 5.4 (with $m = n = 0$). The expressions Λ_1 and Λ_2 remain unchanged when exchanging both pairs $\ell_i \leftrightarrow \ell_j$ and $L_i \leftrightarrow L_j$, so F_1 satisfies property (iii) as well. Moreover,

$$F_2(a,b,c,\ell_1,\ell_2,\ell_3) = F_1(b,c,a,\ell_2,\ell_3,\ell_1)$$

and

$$F_3(a,b,c,\ell_1,\ell_2,\ell_3) = F_1(c,a,b,\ell_3,\ell_1,\ell_2).$$

This can be seen by taking into account the symmetries of Λ_1 and Λ_2 just observed. Hence h satisfies all of the requirements of Lemma 5.4. The same arguments apply when replacing ω_i with $-\omega_i$, so h^{-1} also satisfies all of the requirements of Lemma 5.4. □

Lemma 5.7. *Let $n \in \mathbb{Z}$. Suppose P has angular coordinates $(\alpha_1, \alpha_2, \alpha_3)$, where $\alpha = n(\theta+\pi)/2$. Then angular reflection about P preserves polynomial triangle centers.*

Proof. Let σ denote the angular reflection Eq. (5.1). In angular coordinates, σ is given by

$$\begin{aligned}\psi \mapsto \psi' &= -\psi + n(\theta+\pi)/2 \\ &= -\psi + \theta + (n-2)(\theta+\pi)/2.\end{aligned}$$

By Proposition 3.2, it follows that $\sigma = h^{n-2} \circ s$, where s is isogonal conjugation. By Lemma 5.6, h^{n-2} preserves polynomial triangle centers, so all that remains to be shown is that isogonal conjugation also preserves polynomial triangle centers.

In terms of homogeneous trilinear coordinates, isogonal conjugation is given by $\ell \mapsto \ell^{-1}$ ([1, 273]). This trivially satisfies the conditions of Lemma 5.4. □

Remark 5.8. The angular reflections satisfying the condition in Lemma 5.7 include antigonal conjugation ($n = 0$), isogonal conjugation ($n = 1$), and inversion in the circumcircle ($n = 2$).

Theorem 5.9. *Let h be defined as in Lemma 5.6 and σ any angular reflection satisfying the condition of Lemma 5.7. The group generated by h and σ does not depend on σ, and every plane transformation in the group preserves polynomial triangle centers.*

Proof. This follows immediately from Lemma 5.6 and Lemma 5.7. □

Remark 5.10. The map h^2 is given by $\psi \mapsto \psi + \theta$, which by Proposition 3.2 and Proposition 3.4 implies $h^2 = v \circ s$. The dihedral group of Theorem 3.6, then, is a subgroup of the group in Theorem 5.9. In particular, it too consists of plane transformations which preserve polynomial triangle centers.

5.4. Hofstadter Points

Recall that in the construction described by Theorem 2.11, letting $\psi = r\theta$ for $r \neq 0, 1$ recovers the construction of the Hofstadter r-point H_r in [6]. Let $H^\perp_{r+1/2}$ be defined analogously, but with $\psi = \pi/2 + (r + \frac{1}{2})\theta$. That is, $H^\perp_{r+1/2}$ can be constructed in the same way as the Hofstadter point $H_{r+1/2}$, but with each of the lines AB', AC', BA', BC', CA', and CB' rotated by $\pi/2$.

Lemma 5.11. *H_r has angular coordinates $\psi = r\theta$ and exact tricyclic coordinates*

$$c = R\frac{\sin\left((r-1)\,\theta\right)}{\sin(\theta)}.$$

$H^{\perp}_{r+1/2}$ *has angular coordinates* $\psi = \left(r+\frac{1}{2}\right)\theta + \frac{\pi}{2}$ *and exact tricyclic coordinates*

$$c = R\frac{\cos\left(\left(r-\frac{1}{2}\right)\theta\right)}{\cos\left(\left(r+\frac{1}{2}\right)\theta\right)}$$

Proof. This follows immediately from Theorem 2.11. □

The following theorem appears as a conjecture of Randy Hutson in entry $X(360)$ and $X(5961)$ of [8]. Part (ii) of the theorem is already known, and can be found in [6] without the restriction on r (that is, it is in fact true that H_0 and H_1 are isogonal conjugates).

Theorem 5.12. *Let H_r denote the Hofstadter r-point. Then*
(i) *The inverse-in-circumcircle of H_r is H_{2-r} when $r \neq 0, 1, 2$,*
(ii) *The isogonal conjugate of H_r is H_{1-r} when $r \neq 0, 1$,*
(iii) *The antigonal conjugate of H_r is H_{-r} when $r \neq -1, 0, 1$.*

Proof. By Lemma 5.11, H_r has angular coordinates $\psi = r\theta$. By Proposition 3.4, the inverse of H_r has angular coordinates

$$\psi = -r\theta + 2\theta = (2-r)\theta.$$

By Proposition 3.2, the isogonal conjugate of H_r has angular coordinates

$$\psi = -r\theta + \theta = (1-r)\theta.$$

By Proposition 3.1, the antigonal conjugate of H_r has angular coordinates

$$\psi = -r\theta.$$

By Lemma 5.11, The right-hand sides match the angular coordinates of H_{2-r}, H_{1-r}, and H_r, respectively, provided the listed constraints on r are observed. □

Theorem 5.13. *The maps h and $\rho = h^2$, where h is defined as in Lemma 5.6, act on the points H_r and $H^{\perp}_{r+1/2}$ as in Fig. 9. Note that ρ is isogonal conjugation followed by inversion (cf. Remark 5.10).*

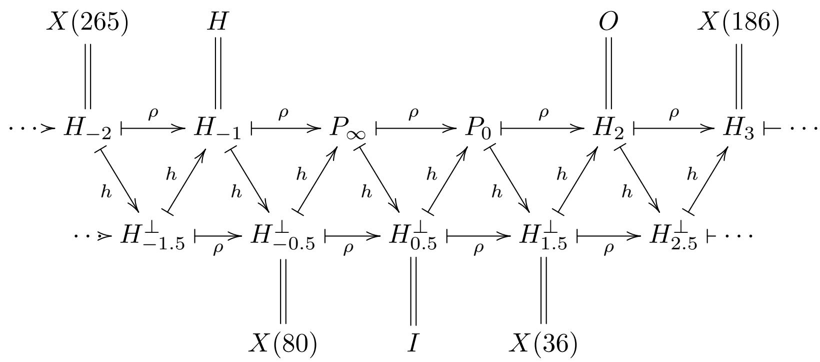


FIGURE 9. The maps h and ρ act on Hofstadter points.

Proof. The map h is defined in angular coordinates by $\psi \mapsto \psi + (\pi + \theta)/2$. As described in Theorem 4.3, the point P_∞ has angular coordinates $\psi = 0$. Hence $h^n(P_\infty)$ has angular coordinates $\psi = (n/2)(\pi + \theta)$. When $n = 2r$, this becomes $\psi = r\theta$. Therefore $h^{2r}(P_\infty) = H_r$, provided that $r \neq 0, 1$. Similarly, when $n = 2r + 1$, $\psi = \left(r + \frac{1}{2}\right)\theta + \frac{\pi}{2}$. So $h^{2r+1}(P_\infty) = H^\perp_{r+1/2}$. □

Note that H_0 and H_1 are missing from the diagram in Theorem 5.13, and that these are exactly the "special" Hofstadter points which are obtained as a limit of other Hofstadter points as $r \to 0$ and $r \to 1$.

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