# RELATED SOLUTIONS TO THE PERSPECTIVE THREE-POINT POSE PROBLEM

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Abstract: The Perspective Three-Point Pose Problem (P3P) involves solving Grunert's system of quadratic equations for the distances from the center of prospective to the three control points, typically resulting in multiple mathematical solutions. Relationships between the corresponding possible camera positions in space have only sparcely been studied. Several efforts have been made though to understand the number of solution points using various assumptions. In this article, the number of solutions is determined in the limiting case where the center of perspective is far from the plane containing the control points, as compared with its distance to the danger cylinder. Moreover, concise formulas are given for the other solutions based on a knowledge of one of the solutions. It turns out that the projection onto the control points plane of the various solution points lie at the intersection points of two rectangular hyperbolas. A certain deltoid curve also plays a crucial role.

## 1 INTRODUCTION

When a camera image includes three points that correspond to three points in space with known positions, the position of the camera can nearly be deduced from this scant information. The mathematical details for doing so have long been understood, but unfortunately, solving the relevant system of equations can result in up to four solutions, only one of which is the actual position of the camera. In this setup, the three known points are referred to as "control points," the camera's optical center (assuming a pinhole camera model here) is the "center of perspective," and the system of equations is "Grunert's system." Assuming, as we will, that the three control points are not collinear, they will instead lie an a particular circle. By extending this circle in the direction perpendicular to the plane containing the control points, one obtains the "danger cylinder," which has importance in the analysis of the problem, known as the "Perspective 3-point pose problem (P3P)".

The issue of repeated solutions to Grunert's system and the weaker circumstance of repeated roots to Grunert's quartic polynomials has been explored by a number of researchers in recent years. There has been a good deal of interest too in determining the number of positive real solutions to the system, based on specific values of its parameters. (Wolfe et al., 1991) provides some excellent geometric insight into these matters by examining various configurations of triangles. A sufficient condition for four positive solutions is given in (Zhang and Hu, 2005). In (Zhang and Hu, 2006) the same authors explore the danger cylinder and use a certain Jacobian determinant to establish that this is where repeated solutions to Grunert's system occur. More recently, (Rieck, to appear) precisely determines the other two solutions when a double solution (necessarily on the danger cylinder) is specified.

(Gao et al., 2003) solves the difficult problem of classifying the number of real solutions and the number of positive solutions, depending on the values of the parameters. Unfortunately this work does not provide much geometric insight into this issue. (Tang et al., 2008) gives a better geometric sense of some of the conditions. (Yang, 1998) and (Faugère et al., 2008) provide algorithms that can assist in the same classification problem.

(Rieck, 2014) is an attempt to move away from the direct study of Grunert's system, involving various distances, and focuses more on the position of the center of perspective in relation to the control points. This has the potential of eventually answering various P3P questions in a satisfying geometric manner. In the present article, its results will be used to obtain a complete understanding of the solutions when the center of prospective is sufficiently far from the plane containing the control points, as compared to its distance from the danger cylinder.

## 2 PROBLEM STATEMENT

The assumptions and notation of (Rieck, 2014) will be used throughout. The basic setup is as follows. It is assumed that a coordinate system is chosen for which the three control points, $P_1$, $P_2$, $P_3$, lie on the unit circle centered about the origin in the $xy$-plane. The danger cylinder is thus given by the equation

$$x^2 + y^2 = 1.$$

For $j = 1,2,3$, let

$$(x_j, y_j, 0) = (\cos\phi_j, \sin\phi_j, 0)$$

be the coordinates of $P_j$, with $-\pi < \phi_j \leq \pi$. Let

$$t_j = \tan(\phi_j/2) = y_j/(1+x_j),$$

so that

$$x_j = (1-t_j^2)/(1+t_j^2)$$

and

$$y_j = 2t_j/(1+t_j^2).$$

For $j = 1,2,3$, let $d_j$ be the distance between the two control points other than $P_j$. The center of perspective will be denoted $p$, and $r_j$ will be the distance from $p$ to $P_j$ ($j = 1,2,3$). The (unknown) coordinates of $p$ will just be denoted $(x,y,z)$. For $j = 1,2,3$, let $\theta_j$ be the angle at $p$ created by the two rays to the two control points other than $P_j$. These angles are presumed to be known since they are easily computed from the camera images of the control points and the camera intrinsics. Let $c_j = \cos\theta_j$.

Finding the coordinates $(x,y,z)$ of the center of prospective $p$ now becomes a matter of solving a pair of systems of equations, as follows:

$$\begin{cases} r_2^2 + r_3^2 - 2c_1 r_2 r_3 = d_1^2 \\ r_3^2 + r_1^2 - 2c_2 r_3 r_1 = d_2^2 \\ r_1^2 + r_2^2 - 2c_3 r_1 r_2 = d_3^2 \end{cases} \quad (1)$$

$$\begin{cases} (x-x_1)^2 + (y-y_1)^2 + z^2 = r_1^2 \\ (x-x_2)^2 + (y-y_2)^2 + z^2 = r_2^2 \\ (x-x_3)^2 + (y-y_3)^2 + z^2 = r_3^2 \end{cases} \quad (2)$$

Since finding the unknowns $r_1$, $r_2$, and $r_3$ serves only an intermediary role in the actual goal of determining $(x,y,z)$, one might consider using classical methods to eliminate them. Without care though, this can produce some very complicated equations. However, in (Rieck, 2014), a few reasonable and interesting equations were discovered that relate $x$, $y$ and $z$ directly to the known parameters $d_1$, $d_2$, $d_3$, $c_1$, $c_2$, and $c_3$.

We now turn our attention to the problem of determining the other solution points if we assume that a particular solution point is known. This will be our focus henceforth. Let us use $(X,Y,Z)$ to denote the coordinates of this given solution point $P$, and let $(x,y,z)$ denote the coordinates of some other solution point $p$, for the same parameter values. Though the parameters $d_1$, $d_2$, $d_3$, $c_1$, $c_2$, and $c_3$ are fixed, the different solution points will have different values for $r_1$, $r_2$, and $r_3$. However. we will soon be ignoring these intermediary quantities.

The stated general problem still seems unwieldy at present. So instead we will consider this problem only when the point $P$ is sufficiently far from the plane containing the control points, as compared with its distance to the danger cylinder. More precisely, we will suppose that the quantity $|1 - X^2 - Y^2|/Z^2$ is sufficiently small. We will obtain precise and interesting formulas for $(x,y)$ in the limit as $|1 - X^2 - Y^2|/Z^2 \to 0$. Though this is a limiting setup, it will nevertheless shed considerable light on setups where $|1 - X^2 - Y^2|/Z^2$ is merely "reasonably" small.

## 3 PROBLEM SOLUTION

Henceforth $(X,Y,Z)$ will be called the "reference solution point," and again, it is presumed to be known. The other solution points, for the same parameters ($d_1$, $d_2$, $d_3$, $c_1$, $c_2$, and $c_3$), will be called "related solutions." We will use $(x,y,z)$ to generically refer to the coordinates of one of these points. However, since it is of no consequence, we will always assume that $Z \geq 0$ and $z \geq 0$. We will be especially interested in the case where $Z$ is large, and will require the following.

**Lemma 1.** *With $X$ and $Y$ fixed, if we change $Z$ by allowing it to grow without bound, then $z/Z$ approaches one in the limit.*

*Proof.* By Lemma 3 in (Rieck, 2014), the quantity $1 - c_1^2 - c_2^2 - c_3^2 + 2c_1c_2c_3$ equals

$$\frac{(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)^2 z^2}{\prod_{j=1}^{3} [(x - x_j)^2 + (y - y_j)^2 + z^2]},$$

and also equals

$$\frac{(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2)^2 Z^2}{\prod_{j=1}^{3} [(X - x_j)^2 + (Y - y_j)^2 + Z^2]}.$$

As $Z$ increases, this tends to zero. Therefore $z$ must also increase so that $z/Z \to 1$, or else $z$ must also tend to zero. We need to discount the latter possibility.

Observe that $1 - c_1^2$ equals

$$\frac{[(y_3-y_2)x+(x_2-x_3)y+(x_3y_2-x_2y_3)]^2 + [(x_2-x_3)^2+(y_2-y_3)^2]\, z^2}{\prod_{j=2}^{3} [(x - x_j)^2 + (y - y_j)^2 + z^2]},$$

and also equals

$$\frac{[(y_3-y_2)x+(x_2-x_3)y+(x_3y_2-x_2y_3)]^2 + [(x_2-x_3)^2+(y_2-y_3)^2]\, Z^2}{\prod_{j=2}^{3} [(X - x_j)^2 + (Y - y_j)^2 + Z^2]}.$$

To see this, first solve (1) for $c_1$, and then check that $1 - c_1^2 = -([(r_2 - r_3)^2 - d_1^2][(r_2 + r_3)^2 - d_1^2]) / (4r_2^2r_3^2)$ $= (2r_2^2r_3^2 + 2d_1^2r_2^2 + 2d_1^2r_3^2 - r_2^4 - r_3^4 - d_1^4) / (4r_2^2r_3^2)$. Then make further substitutions and simplify.

As $Z \to \infty$, $1 - c_1^2$ tends to zero. Now suppose $z \to 0$ too. This forces $(y_3 - y_2)x + (x_2 - x_3)y + (x_3y_2 - x_2y_3) = 0$. That is, the point $(x, y)$ in the $xy$-plane must be on the line that connects the control points $P_2$ and $P_3$. By similar reasoning concerning $c_2$ and $c_3$, $(x, y)$ must also be on the line connecting $c_3$ and $c_1$, as well as the line connecting $c_1$ and $c_2$. This is impossible. So $z$ cannot be approaching 0 as $Z \to \infty$.

□

Theorem 1 of (Rieck, 2014) asserts that a certain quantity that depends only on the parameters $d_1$, $d_2$, $d_3$, $c_1$, $c_2$, and $c_3$, also equals

$$A(\phi_1, \phi_2, \phi_3; x, y) + B(\phi_1, \phi_2, \phi_3; x, y)(1 - x^2 - y^2)/z^2, \quad (3)$$

where $A(\phi_1, \phi_2, \phi_3; x, y)$ and $B(\phi_1, \phi_2, \phi_3; x, y)$ are certain quadratic polynomials in $x$ and $y$. Here we can of course replace $(x, y, z)$ with $(X, Y, Z)$. Fixing $(X, Y)$, as we let $Z \to \infty$, the above quantity approaches $A(\phi_1, \phi_2, \phi_3; X, Y)$. By Lemma 1, $z \to \infty$ too, so the same quantity also approaching $A(\phi_1, \phi_2, \phi_3; x, y)$. For large $Z$, we thus have

$$A(\phi_1, \phi_2, \phi_3; x, y) \approx A(\phi_1, \phi_2, \phi_3; X, Y), \quad (4)$$

where "$\approx$" becomes equality in the limit.

One further assumption will prove useful, though it was not used in (Rieck, 2014). Henceforth, we will assume that $\phi_1 + \phi_2 + \phi_3 = 0$. This imposes no serious constraint since our chosen coordinate system can always be rotated to make this so. This substantially simplifies the equations and associated geometric interpretation.

**Lemma 2.** *Assuming that $\phi_1 + \phi_2 + \phi_3 = 0$, and $\phi_3 \neq 0$, the equation*

$$A(\phi_1, \phi_2, \phi_3; x, y) = A(\phi_1, \phi_2, \phi_3; X, Y)$$

*becomes simply*

$$2\,(1 + x_3)\,(y + xy - Y - XY) = y_3\,(x^2 - 2x - y^2 - X^2 + 2X + Y^2). \quad (5)$$

*Proof.* Using Theorem 1 and the last part of Lemma 3 in (Rieck, 2014), and after making the substitution

$$t_3 \to (t_1 + t_2)/(t_1t_2 - 1)$$

(because $\phi_3 = -\phi_1 - \phi_2$), and simplifying, one obtains $A(\phi_1, \phi_2, \phi_3; x, y) =$

$$\frac{(t_1 + t_2)(3t_1^2t_2^2 - t_1^2 - t_2^2 - 8t_1t_2 + 3)}{(t_2 - t_1)(1 + t_1^2)(1 + t_2^2)} + \frac{(t_1 + t_2)(y^2 - x^2 + 2x) + 2(t_1t_2 - 1)(x + 1)y}{t_2 - t_1}.$$

This remains so when $(x, y)$ is replaced with $(X, Y)$. The difference between these equals

$$\frac{(t_1 + t_2)(y^2 - x^2 + 2x - Y^2 + X^2 - 2X)}{t_2 - t_1} + \frac{2(t_1t_2 - 1)(xy + y - XY - Y)}{t_2 - t_1}.$$

This equals

$$\frac{t_3}{x_1 - x_2}\,\big[\,y_3\,(y^2 - x^2 + 2x - Y^2 + X^2 - 2X) + 2\,(1 + x_3)\,(xy + y - XY - Y)\,\big],$$

which follow by substituting $x_3 \to (1-t_3^2)/(1+t_3^2)$ and $y_3 \to 2t_3/(1+t_3^2)$, and then applying the above substitution for $t_3$. This yields the claim in the lemma. Note that $t_3/(x_1-x_2) = (1+t_1^2)(1+t_2^2)/[2(t_2-t_1)(t_1t_2-1)]$, which cannot equal zero, though it is undefined if $\phi_3 = 0$.

□

**Lemma 3.** *Assume that $\phi_1+\phi_2+\phi_3 = 0$, $\phi_1 \neq 0$, $\phi_2 \neq 0$ and $\phi_3 \neq 0$. In the limiting case where $Z \to \infty$ (holding X and Y fixed), equations (5) through (9) all hold. Additionally, if $(x,y)$ and $(X,Y)$ are distinct points, then (10) and (11) hold as well.*

*Proof.* By symmetry, equation (5) gives rise to two other similar equations, as follows:

$$2(1+x_1)(y+xy-Y-XY) = y_1(x^2-2x-y^2-X^2+2X+Y^2) \quad (6)$$

and

$$2(1+x_2)(y+xy-Y-XY) = y_2(x^2-2x-y^2-X^2+2X+Y^2). \quad (7)$$

Taking a linear combination of these yields

$$(y_1+x_2y_1-y_2-x_1y_2)(y+xy-Y-XY) = 0.$$

But

$$y_1+x_2y_1-y_2-x_1y_2 = \frac{4(t_1-t_2)}{(1+t_1^2)(1+t_2^2)} \neq 0,$$

and hence we must conclude that

$$(x+1)y = (X+1)Y. \quad (8)$$

Then from (5), (6) and (7), we can also deduce that

$$(x-1)^2-y^2 = (X-1)^2-Y^2. \quad (9)$$

Eliminating $y$ from (8) and (9) yields the resultant polynomial $(x-X)\cdot$

$$[x^3+Xx^2+(Y^2+2X-3)x+(XY^2+2Y^2+X-2)].$$

This must vanish, and so if $x \neq X$, we obtain

$$x^3+Xx^2+(Y^2+2X-3)x+(XY^2+2Y^2+X-2) = 0. \quad (10)$$

By instead eliminating $x$ from (8) and (9), and assuming $y \neq Y$, we likewise obtain

$$y^3+Yy^2+(X+1)(X-3)y+(X+1)^2Y = 0. \quad (11)$$

Concerning the claim in the lemma about (10) and (11), consider first the case where $X \neq -1$ and $Y \neq 0$. Equation (8) here implies that $X = x$ if and only if $Y = y$. So under the assumption that $(X,Y)$ and $(x,y)$ are distinct, (10) and (11) hold.

Next, consider the two special cases where $X = \pm 1$ and $Y = 0$. Here (10) can be seen to hold if $x = X$, and as has already been observed, it must hold when $x \neq X$. Likewise (11) must hold whether or not $y = Y$.

Lastly, consider the case where $X = -1$ or $Y = 0$, but $X^2+Y^2 \neq 1$, and again assume that $(X,Y)$ and $(x,y)$ are distinct. We can consider making infinitesimal changes to $c_1$, $c_2$ and $c_3$, causing corresponding infinitesimal changes to $(X,Y,Z)$ and $(x,y,z)$. Because $(X,Y,Z)$ is not on the danger cylinder, the mapping between $(c_1,c_2,c_3)$ and $(X,Y,Z)$ is locally invertible (cf. (Rieck, 2014) and (Zhang and Hu, 2006)). Thus the infinitesimal changes can be made so as to cause $X \neq -1$ and $Y \neq 0$, and it will keep $(X,Y)$ and $(x,y)$ distinct. By the first case, (10) and (11) hold for these new points, but then by continuity, (10) and (11) must also hold for the original points.

The above three cases cover all the possibilities for which $(X,Y)$ and $(x,y)$ are distinct points. So the claim stated in the lemma concerning (10) and (11) is true.

□

Notice that (8) and (9) mean that the reference point and all of its related solution points project onto the $xy$-plane at the points of intersection of two rectangular hyperbolas.

**Theorem 1.** *Assume that $\phi_1+\phi_2+\phi_3 = 0$, $\phi_1 \neq 0$, $\phi_2 \neq 0$ and $\phi_3 \neq 0$. Given a reference solution point $(X,Y,Z)$, let $\Delta =$*

$$27-24XY^2+8X^3-18(X^2+Y^2)-(X^2+Y^2)^2.$$

*Let $\sqrt{-3\Delta}$ be either of the (complex) square roots of $-3\Delta$. Let*

$$\Gamma = (X-3)^3+9(X+3)Y^2 + 3\sqrt{-3\Delta}\,Y.$$

*Let $\sqrt[3]{\Gamma}$ be any of the three (complex) cube roots of $\Gamma$. Then taking*

$$x = -\frac{1}{3}\left[X+\sqrt[3]{\Gamma}+\frac{(X-3)^2-3Y^2}{\sqrt[3]{\Gamma}}\right]$$

*and*

$$y = \frac{(X+1)Y}{x+1}$$

*yields the first two coordinates of a related solution point $(x,y,z)$. However, the coordinates might not be real numbers. Conversely, the first two coordinates of each related solution point can be obtained in this manner.*

*Proof.* The left side of (10) is a cubic polynomial in $x$, $x^3+bx^2+cx+d$, with $b=X$, $c=Y^2+2X-3$ and $d=XY^2+2Y^2+X-2$. Following a classical technique for finding its roots, set $\Delta_0=b^2-3ac=(X-3)^2-3Y^2$, and set $\Delta_1=2b^3-9bc+27d=2[(X-3)^3+9(X+3)Y^2]$. Now, $\Delta_1^2-4\Delta_0^3=-108Y^2\Delta$. So $\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}=2\Gamma$. The roots of the cubic are thus $-(b+\sqrt[3]{\Gamma}+\Delta_0/\sqrt[3]{\Gamma})/3$, by a well-known formula for the roots of cubics (essentially Cardano's formula). The roots are the first coordinates of the related solution point. The matching second coordinates can be obtained using (8). □

Note: The equation $\Delta=0$ describes the same deltoid (hypocycloid) curve that was introduced in (Rieck, to appear).

After using Theorem 1 to solve for $x$ and $y$, the remaining coordinate $z$ can be determined from the systems (1) and (2), as follows. Using (2), each $r_j^2$ can be written as a linear function of $z^2$ with known coefficients. Using (1), each $c_j^2$ is equal to a rational function whose numerator and denominator are known quadratic functions of $z^2$. Since the values of the $c_j$ are known, this leads to three quadratic equations in $z^2$ with known coefficients. Taking any two of these, one can then reduce these to a linear equation in $z^2$ with known coefficients, which can then be solved for $z^2$.

**Corollary 1.** *Assume the reference point $P$ has real coordinates $(X,Y,Z)$. Given a related solution point $(x,y,z)$, let*

$$\delta=27-24xy^2+8x^3-18(x^2+y^2)-(x^2+y^2)^2.$$

*In the limiting case where $Z\to\infty$ (with $X$ and $Y$ fixed), there are three possibilities:*

1. *If $\Delta>0$, then $P$ has three distinct real-valued related solution points, and each of these satisfies $\delta>0$.*
2. *If $\Delta=0$, then $P$ has three real-valued related solution points, and each of these satisfies $\delta\geq 0$, but at least two of them coalesce to form a repeated solution point.*
3. *If $\Delta<0$, then $P$ has exactly one real-valued related solution points, and it satisfies $\delta<0$.*

*Proof.* As a cubic polynomial in $x$, the discriminant of (10) is $4Y^2\Delta$. Likewise, as a cubic polynomial in $y$, the discriminant of (11) is $4(X+1)^2\Delta$. Since the discriminant of (10) is $4Y^2\Delta$, it has three district real roots when $\Delta>0$. If $\Delta=0$, it still has only real roots but at least two of these coalesce, resulting in at most two distinct roots. If $\Delta<0$, (10) has only one real root. For each root of (10), there is a matching root of (11) that can be found by simply using (8). □

Visual simulations exploring various positions of the reference point and corresponding related points suggests the following.

**Conjecture 1.** *Assume the reference point $P$ has real coordinates $(X,Y,Z)$. There are the following five possibilities:*

1. *If $X^2+Y^2<1$, then there are three real-valued related solution points $(x,y,z)$, each with $x^2+y^2>1$ and $\delta>0$.*
2. *If $X^2+Y^2=1$, then there are three real-valued related solution points, but at least one of these coalesces with the reference point, and except when $(X,Y)$ is one of three particular points, exactly one related solution point does so, and the other two satisfy $\delta=0$.*
3. *If $X^2+Y^2>1$ and $\Delta>0$, then there are three real-valued related solution points, each satisfying $\delta>0$. One of these satisfies $x^2+y^2<1$ and the other two satisfy $x^2+y^2>1$.*
4. *If $X^2+Y^2>1$ and $\Delta=0$, then there are three real-valued related solution points, except that at least two of them coalesce and satisfy $x^2+y^2=1$. If exactly two of them coalesce, then the other one satisfies $x^2+y^2>1$ and $\delta=0$.*
5. *If $\Delta<0$, then there is only one real-valued related solution point, and it satisfies $\delta<0$.*

# 4 CONCLUSION

Together, Theorem 1, Corollary 1 and Conjecture 1 gives a very complete description of the relationship between the solution points for the limiting case examined here. The results in this article provide useful insights that can be carried back to gain a greater understanding of the general situation for P3P. Some simulations have demonstrated that as long as the reference point avoids getting close to the control points plane, the distribution of the related points will be similar to that of the limiting case.

In continuing the analysis of this article, the next step would seem to be to remove the restriction that $|1-x^2-y^2|/z^2$ be negligible. This would require the inclusion of the "B part" of the formula in Theorem 1 of (Rieck, 2014). There is little doubt that this would result in significantly more complicated relationships among the solution points. There is however a reasonable hope that some compelling interplay between the solution points will be discovered. Ideally, eventually, a good geometric understanding of all the salient aspects of P3P will emerge.

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